A low–order Local Projection method for the incompressible Navier–Stokes equations in two- and three-dimensions

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This work proposes and analyzes a new Local Projection Stabilized finite element method (LPS for short) for the non-linear incompressible Navier–Stokes equations. Stokes problems defined element–wisely drive the construction of the stabilized terms which make the present method stable for continuos velocity and (dis)continuos pressure finite element pairs $\mathbb{P}_1 \times \mathbb{P}_l$, $l = 0, 1$, in two- and three -dimensions. Existence and uniqueness of a discrete solution and a non–singular branch of solutions are proved under standard assumptions. Also, we establish that the LPS method achieves optimal error estimates in the natural norms. Numerics assess the theoretical results and validate the LPS method in the three-dimensional case.

Keywords: Navier–Stokes equations, stabilized finite element, low–order method

1. Introduction

Stabilized finite element methods have been used extensively to solve the incompressible Navier–Stokes equations since the seminal paper Brooks & Hughes (1982), followed by the works Franca & Frey (1992) and Lube & Tobiska (1990). Introduced as a strategy to circumvent (unstable) mixed finite elements, these methods make the equal order and the simplest elements (i.e., the space pair $\mathbb{P}_1 \times \mathbb{P}_0$) inf–sup stable (see Ern & Guermond (2004) and Girault & Raviart (1986) for details) by adding extra variational terms to the classical Galerkin method. Originally, stabilized methods add residual local terms to the Galerkin formulation preserving consistency. Classical examples are the SUPG, GLS and SDFEM methods Franca *et al.* (1992); Baiocchi *et al.* (1993); Tobiska & Verfürth (1996), just to cite a few among the vast literature on the subject.

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Recently, a family of non-residual stabilized finite element method have become popular as they add fewer (symmetric) terms to the formulation when compared to their residual counterpart. Such schemes, originally proposed in Becker & Braack (2001) for the Stokes problem (see also Codina (2008), Burman *et al.* (2006), Braack & Lube (2009), Ganesan & Tobiska (2010) for alternative versions) and known as Local Projection Stabilized (LPS) methods, may be seen as a "term-by-term" stabilized method for which the extra terms are based on the projection onto a polynomial space defined with respect to a coarser triangulation through a de-refinement of the original mesh. Hence, in general, such methods demand a special data structure to be implemented. Meanwhile, the so-called Residual Local Projection (RELP) stabilized methods Barrenechea & Valentin (2010a) reincorporate the residual idea into the construction of stabilized methods, but now merged with the fundamental idea behind the LPS methodology. This is achieved through interpolation operators (fluctuation operators) defined on finite dimensional spaces generated by the solution of local problems (see Barrenechea & Valentin (2010b) for this idea applied to the Oseen equation). From this point of view, the RELP method resembles to the Residual-Free-Bubble (RFB) approach Brezzi *et al.* (1998) which introduced local driven problems to build stabilized terms without priori knowledge on their structures. In Araya *et al.* (2012), the original RELP method given in Barrenechea & Valentin (2010b) was extended to the incompressible Navier–Stokes equations, and proved to be well-posed and optimal convergent.

In this work, we propose and analyze a new LPS method for the incompressible Navier-Stokes equations built up on the RELP method given in Araya *et al.* (2012). This leads to a much simpler method (in terms of number of stabilized terms) than the original RELP method. Here, we focus on the piecewise-linear continuos space for the velocity variable, and the piecewise constant or linear (dis)continuos spaces for the pressure, as those choices are the most common in practice. Also, these choices make closed formula available to approximating the second level problems without undermining theoretical results. As a result, the proposed stabilized method may be implemented easily in standard finite element codes.

In regard to theoretical results, this work establishes well–posedness and an optimal convergence result in the natural norms for the new LPS method within the framework presented in Araya *et al.* (2012) and Tobiska & Verfürth (1996). Specifically, the stability and convergence analysis rely on the fixed point theory proposed in Brezzi *et al.* (1980); Girault & Raviart (1986). This point of view has also been adopted in Tobiska & Verfürth (1996) to analyze the stabilized method originally proposed in Franca & Frey (1992) (see also Codina & Blasco (2000) for a related result). Due to the particular structure of the method, the proof requests the construction of a new stabilized finite element method for the Stokes equation, which is also analyzed. It is worth mentioning that a detailed analysis for the three-dimensional case is performed under standard conditions. This differs from Araya *et al.* (2012). Also interesting, we show that the underlined lack of consistency of the LPS method (which is absent in Araya *et al.* (2012)) remains controlled, i.e., it stays at order of the leading errors. The stated theoretical results are assessed through a large variety of three-dimensional computational benchmarks.

The paper is outlined as follows: In Section 2 we present some notations, definitions and technical results which will be used throughout this work. The LPS method is introduced in Section 3 as well as existence and uniqueness results. Section 4 is devoted to the *a priori* error analysis and numerical validations are presented in Section 5. Conclusions are given in Section 6.

2. Notations and preliminary results

Let $\Omega \subseteq \mathbb{R}^d$, $(d=2 \text{ or } 3)$ be a polygonal (polyhedral) open domain. The steady incompressible Navier–Stokes equations consists of finding the velocity and the pressure $(\boldsymbol{u},\tilde{p})$ such that

$$\begin{aligned} -\nu\,\Delta\boldsymbol{u} + (\nabla\boldsymbol{u})\,\boldsymbol{u} + \nabla\tilde{p} = \tilde{\boldsymbol{f}}, \quad \nabla\cdot\boldsymbol{u} = 0 \qquad &\text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\nu \in \mathbb{R}^+$ is the fluid viscosity and $\tilde{\boldsymbol{f}} \in L^2(\Omega)^d$. Adopting standard notations for Sobolev spaces, the weak form associated to (2.1) reads: *Find* $(\boldsymbol{u},\tilde{p}) \in \boldsymbol{H}\times Q := H_0^1(\Omega)^d \times L_0^2(\Omega)$ *such that:*

$$\nu\,(\nabla\boldsymbol{u},\nabla\boldsymbol{v}) + ((\nabla\boldsymbol{u})\,\boldsymbol{u},\boldsymbol{v}) - (\tilde{p},\nabla\cdot\boldsymbol{v}) + (q,\nabla\cdot\boldsymbol{u}) = (\tilde{\boldsymbol{f}},\boldsymbol{v}) \quad \text{for all } (\boldsymbol{v},q) \in \boldsymbol{H}\times Q,$$

where $(\cdot,\cdot)$ stands for the $L^2(\Omega)$-inner product (or $L^2(\Omega)^d$).

Let D be an open subset of Ω, we denote by $\|\cdot\|_{m,D}$ the norm in $H^m(D)$, and by $\|\cdot\|_{m,q,D}$ the norm in $W^{m,q}(D)$ with $m \geqslant 0$ and $1 \leqslant q \leqslant \infty$. We denote, as usual, $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$ equipped with the dual norm $\|\cdot\|_{-1,\Omega}$ and the duality product $\langle\cdot,\cdot\rangle$, and $H^0(\Omega) = L^2(\Omega)$ and $W^{0,q}(\Omega) = L^q(\Omega)$. Also, we equip the space $\boldsymbol{H}\times Q$ with the norm $|\!|\!|\cdot|\!|\!|$ given as follows

$$|\!|\!|(\boldsymbol{v},q)|\!|\!| := \left\{|\boldsymbol{v}|_{1,\Omega}^2 + \|q\|_{0,\Omega}^2\right\}^{1/2},$$

and the dual space of $\boldsymbol{H}\times Q$ (denoted by $(\boldsymbol{H}\times Q)'$) with the norm $\|\cdot\|_{(\boldsymbol{H}\times Q)'}$ given by

$$\|(\boldsymbol{v},q)\|_{(\boldsymbol{H}\times Q)'} := \sup_{|\!|\!|(\boldsymbol{w},r)|\!|\!|\leqslant 1} \left\{\langle\boldsymbol{v},\boldsymbol{w}\rangle + (q,r)\right\}.$$

Let $\{\mathscr{T}_h\}_{h>0}$ be a family of regular triangulations of $\bar{\Omega}$, built up of triangles $(d=2)$, or tetrahedra $(d=3)$ K with boundary ∂K and characteristic length $h_K := \mathrm{diam}(K)$. For each triangulation $\mathscr{T}_h$, we define $h := \max\{h_K : K \in \mathscr{T}_h\}$, and denote by $\mathscr{E}_h$ the set of internal edges (faces) F of the partition $\mathscr{T}_h$, and $h_F := \mathrm{diam}(F)$. We denote by $\boldsymbol{n}$ the outward normal vector on ∂K; $[\![v]\!]$ stands for the jump of v across F. In addition, for $K \in \mathscr{T}_h$ and $F \in \mathscr{E}_h$, we define the neighborhoods $\omega_K := \{K' \in \mathscr{T}_h : K' \cap K \neq \emptyset\}$ and $\omega_F := \{K \in \mathscr{T}_h : F \cap K \neq \emptyset\}$. Finally, we denote by Π_S, with $S \subset \mathbb{R}^d$, the orthogonal projection onto the constant space $\mathbb{P}_0(S)$, i.e., $\Pi_S(q) := \frac{(q,1)_S}{|S|}$, and by $H^m(\mathscr{T}_h)$, $m \geqslant 1$ the space of functions in $L^2(\Omega)$ whose restriction to each $K \in \mathscr{T}_h$ belongs to $H^m(K)$.

We associate to the partition $\mathscr{T}_h$ the discrete space for the velocity $\boldsymbol{H}_h$, composed of vector-valued piecewise linear continuous functions with zero trace on $\partial\Omega$. To approximate the pressure, we adopt the discrete space Q_h defined as the space of piecewise polynomial functions of degree l, $(l=0,1)$ with zero mean value on Ω. In the case $l=1$, the space Q_h may represent continuous or discontinuous functions.

Next, we equip the dual of the discrete spaces with the following norm

$$\|(\boldsymbol{v},q)\|_{(\boldsymbol{H}_h\times Q_h)'} := \sup_{|\!|\!|(\boldsymbol{w}_h,r_h)|\!|\!|\leqslant 1} \left\{\langle\boldsymbol{v},\boldsymbol{w}_h\rangle + (q,r_h)\right\}.$$

Also, the differential of a mapping $F : \boldsymbol{H}\times Q \to \boldsymbol{H}\times Q$ with respect to $(\boldsymbol{u},q)$ is denoted by $D_{\boldsymbol{u},p}F(\boldsymbol{v},q) \in \mathscr{L}(\boldsymbol{H}\times Q)$, where $\mathscr{L}(\boldsymbol{H}\times Q)$ stands for the space of linear mappings acting on elements of $\boldsymbol{H}\times Q$ with values in $\boldsymbol{H}\times Q$ and equipped with the usual norm $\|\cdot\|_{\mathscr{L}(\boldsymbol{H}\times Q)}$.

Now, we recall some classical results which will be needed in the forthcoming analysis sections.

LEMMA 2.1 There exist positive constants α and β, depending only on Ω, such that for all $\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}$ and $q \in Q$, it holds

$$\sup_{\boldsymbol{v}\in\boldsymbol{H}\setminus\{\boldsymbol{0}\}} \frac{(q,\nabla\cdot\boldsymbol{v})}{|\boldsymbol{v}|_{1,\Omega}} \geqslant \beta\,\|q\|_{0,\Omega}, \tag{2.2}$$

$$((\nabla\boldsymbol{u})\boldsymbol{w},\boldsymbol{v}) \leqslant \alpha\,|\boldsymbol{u}|_{1,\Omega}|\boldsymbol{w}|_{1,\Omega}|\boldsymbol{v}|_{1,\Omega}. \tag{2.3}$$

Moreover, for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}$, it holds

$$((\nabla\boldsymbol{v})\boldsymbol{w},\boldsymbol{v}) = -\frac{1}{2}(\nabla\cdot\boldsymbol{w},\boldsymbol{v}\cdot\boldsymbol{v}). \tag{2.4}$$

Proof. See Girault & Raviart (1986) and Temam (1995). □

The following inverse inequality estimates hold for the functions in $\boldsymbol{H}_h$ and Q_h.

LEMMA 2.2 There exists a positive constant C, independent of h, such for all $\boldsymbol{v}_h \in \boldsymbol{H}_h$ and all $q_h \in Q_h$, we have

$$\|\boldsymbol{v}_h\|_{l,p,K} \leqslant C\,h_K^{m-l+d(\frac{1}{p}-\frac{1}{q})}\|\boldsymbol{v}_h\|_{m,q,K}, \tag{2.5}$$

$$\|[\![q_h]\!]\|_{0,F} \leqslant C\,h_F^{-1/2}\|q_h\|_{0,\omega_F}, \tag{2.6}$$

where $0 \leqslant m \leqslant l$ and $1 \leqslant p,q \leqslant \infty$.

Proof. See Lemma 1.138 in Ern & Guermond (2004). □

Some properties of the orthogonal projection Π_K, onto the constant space, are summarized in the following result.

LEMMA 2.3 There exists a constant $C > 0$, independent of h, such that

$$\|v-\Pi_K v\|_{0,K} \leqslant C\,h_K\,|v|_{1,K} \qquad \forall v \in H^1(K), \tag{2.7}$$

$$\|\Pi_K v\|_{0,K} \leqslant \|v\|_{0,K} \qquad \forall v \in L^2(K). \tag{2.8}$$

Proof. See Proposition 1.135 in Ern & Guermond (2004). □

Let $\mathscr{I}_h : \boldsymbol{H}\cap H^2(\Omega)^d \longrightarrow \boldsymbol{H}_h$ be the Lagrange interpolation operator for the velocity, and let $\mathscr{J}_h : Q \longrightarrow Q_h$ be either a modified Clément operator for the continuous pressures case ($l = 1$) or the orthogonal projection onto the space Q_h in the discontinuous pressure case ($l = 0,1$). These interpolation operators satisfy the following inequalities (see Clément (1975); Ern & Guermond (2004))

$$|\boldsymbol{v}-\mathscr{I}_h\boldsymbol{v}|_{m,K} \leqslant C\,h_K^{2-m}\,|\boldsymbol{v}|_{2,K} \qquad \forall \boldsymbol{v} \in H^2(K)^d, \tag{2.9}$$

$$|\mathscr{I}_h\boldsymbol{v}|_{1,K} \leqslant C\,\|\boldsymbol{v}\|_{2,K} \qquad \forall \boldsymbol{v} \in H^2(K)^d, \tag{2.10}$$

$$\|\boldsymbol{v}-\mathscr{I}_h\boldsymbol{v}\|_{1,r,K} \leqslant C\,\|\boldsymbol{v}\|_{1,r,K} \qquad \forall \boldsymbol{v} \in W^{1,r}(K)^d\,,\ \forall r \in (2,\infty], \tag{2.11}$$

$$\|\boldsymbol{v}-\mathscr{I}_h\boldsymbol{v}\|_{\infty,K} \leqslant C\,\|\boldsymbol{v}\|_{\infty,K} \qquad \forall \boldsymbol{v} \in \mathscr{C}^0(K)^d, \tag{2.12}$$

$$|q-\mathscr{J}_h q|_{i,K} \leqslant C\,h_K^{j-i}\,|q|_{j,\omega_K} \qquad \forall q \in H^j(\omega_K), \tag{2.13}$$

$$\|q-\mathscr{J}_h q\|_{0,F} \leqslant C\,h_F^{j-1/2}\,\|q\|_{j,\omega_F} \qquad \forall q \in H^j(\omega_F), \tag{2.14}$$

where $0 \leqslant m \leqslant 2$, and $0 \leqslant i \leqslant 1$, $1 \leqslant j \leqslant l+1$ and C is a positive constant independent of h.

2.1 *The scaled model and the Stokes case*

The analysis of the well–posedness and the convergence aspects of the present LPS method will rely on the Brezzi, Rappaz and Raviart theory proposed in Girault & Raviart (1986) within the concept of branches of non-singular solutions. Thereby, we assume that there exist neighborhoods in which the solution of the Navier–Stokes equations is unique (i.e., the solutions are isolated). It turns out that this effectively occurs in many practical problems. Moreover, the solution depends continuously with respect to the viscosity. As a result, and since the viscosity belongs to a compact set, each solution of the Navier–Stokes equations generates an isolated branch. Now, it is more convenient to consider the scaled form of (2.1). To this end, we set $\tilde{p} = \nu\, p$, $\tilde{\boldsymbol{f}} = \nu\, \boldsymbol{f}$, and $\lambda = \nu^{-1}$, and rewrite (2.1) as follows

$$\begin{gathered} -\Delta \boldsymbol{u}_\lambda + \lambda\,(\nabla \boldsymbol{u}_\lambda)\,\boldsymbol{u}_\lambda + \nabla p_\lambda = \boldsymbol{f}, \quad \nabla\cdot \boldsymbol{u}_\lambda = 0 \quad \text{in } \Omega, \\ \boldsymbol{u}_\lambda = \mathbf{0} \quad \text{on } \partial\Omega. \end{gathered} \tag{2.15}$$

The standard weak formulation of problem (2.15) is given by: *Find* $(\boldsymbol{u}_\lambda, p_\lambda) \in \boldsymbol{H}\times Q$ *such that*

$$(\nabla \boldsymbol{u}_\lambda, \nabla \boldsymbol{v}) + \lambda\,((\nabla \boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda, \boldsymbol{v}) - (p_\lambda, \nabla\cdot\boldsymbol{v}) + (q, \nabla\cdot\boldsymbol{u}_\lambda) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall (\boldsymbol{v}, q) \in \boldsymbol{H}\times Q. \tag{2.16}$$

We assume in this work that problem (2.16) admits at least one solution, which is unique provided λ is sufficiently small. Next, observe that (2.16) can be written in a operator form as follows

$$F(\lambda, \boldsymbol{u}_\lambda, p_\lambda) := (\boldsymbol{u}_\lambda, p_\lambda) + T\,G(\lambda, \boldsymbol{u}_\lambda, p_\lambda) = \mathbf{0}, \tag{2.17}$$

where $G(\lambda, \boldsymbol{u}_\lambda, p_\lambda) \in \boldsymbol{H}'\times Q$ is given by

$$\langle G(\lambda, \boldsymbol{u}_\lambda, p_\lambda), (\boldsymbol{v}, q)\rangle := \lambda\,((\nabla \boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda, \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall (\boldsymbol{v}, q) \in \boldsymbol{H}\times Q,$$

and $T : \boldsymbol{H}'\times Q \longrightarrow \boldsymbol{H}\times Q$ denotes the Stokes operator, which associates for each $(\boldsymbol{w}, r) \in \boldsymbol{H}'\times Q$, the unique solution $(\boldsymbol{u}, p) \in \boldsymbol{H}\times Q$ of

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \nabla\cdot\boldsymbol{v}) + (q, \nabla\cdot\boldsymbol{u}) = \langle \boldsymbol{w}, \boldsymbol{v}\rangle + (r, q),$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{H}\times Q$.

Using such a viewpoint, we introduce $T_h : \boldsymbol{H}'\times Q \longrightarrow \boldsymbol{H}_h\times Q_h$ the discrete counterpart of the Stokes operator, which associates to each $(\boldsymbol{w}, r) \in \boldsymbol{H}'\times Q$, the unique solution of the following stabilized method: *Find* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h\times Q_h$ *such that*

$$\boldsymbol{B}((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) = \langle \boldsymbol{w}, \boldsymbol{v}_h\rangle + (r, q_h) \qquad \forall (\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h\times Q_h,$$

where the bilinear form $\boldsymbol{B}(\cdot,\cdot)$ is given by

$$\begin{aligned} \boldsymbol{B}((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) \;:=\; & (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - (p_h, \nabla\cdot\boldsymbol{v}_h) + (q_h, \nabla\cdot\boldsymbol{u}_h) \\ & + \sum_{K\in\mathscr{T}_h} \left[\big(\chi_h(p_h), \chi_h(q_h)\big)_K + (\lambda\chi_h(\boldsymbol{x}\nabla\cdot\boldsymbol{u}_h), \lambda\chi_h(\boldsymbol{x}\nabla\cdot\boldsymbol{v}_h))_K \right] \\ & + \sum_{F\in\mathscr{E}_h} \frac{h_F}{12}\,(\llbracket p_h\rrbracket, \llbracket q_h\rrbracket)_F\,. \end{aligned}$$

Here $\chi_h := \mathrm{I} - \Pi_K$ is the so-called *fluctuation* operator.

REMARK 2.1 The discrete Stokes operator T_h induces a new stabilized finite element method which, as far as we are aware, is new in the literature. Thereby, we introduce the following mesh–dependent norm

$$\|(\boldsymbol{v}_h,q_h)\|_h^2 := |\boldsymbol{v}_h|_{1,\Omega}^2 + \sum_{K\in\mathscr{T}_h} \|\chi_h(q_h)\|_{0,K}^2 + \|\lambda\,\chi_h(\boldsymbol{x}\nabla\cdot\boldsymbol{v}_h)\|_{0,K}^2 + \sum_{F\in\mathscr{E}_h} \frac{h_F}{12}\,\|[\![q_h]\!]\|_{0,F}^2 \quad \forall(\boldsymbol{v}_h,q_h)\in\boldsymbol{H}_h\times Q_h,$$

and it is easy to see from the definition of $\boldsymbol{B}$ that it holds

$$\boldsymbol{B}((\boldsymbol{v}_h,q_h),(\boldsymbol{v}_h,q_h)) = \|(\boldsymbol{v}_h,q_h)\|_h^2 \qquad \forall(\boldsymbol{v}_h,q_h)\in\boldsymbol{H}_h\times Q_h,$$

and, then, the operator T_h is well–defined. □

Observe that we can easily adapt the analysis presented in Appendix A of Araya *et al.* (2012) to prove the following estimates.

LEMMA 2.4 There exist constants $C, C' > 0$, independent of h and λ, such that it holds

$$\begin{aligned} |||(T-T_h)(\boldsymbol{w},0)||| &\leqslant C\,h\,(1+\lambda\,h)^2\,\|\boldsymbol{w}\|_{0,\Omega} & \forall\boldsymbol{w}\in L^2(\Omega)^d,\\ |||T_h(\boldsymbol{w},q)||| &\leqslant C'\,(1+\lambda\,h)^2\,\|(\boldsymbol{w},q)\|_{(\boldsymbol{H}_h\times Q_h)'} & \forall(\boldsymbol{w},q)\in(\boldsymbol{H}\times Q)'. \end{aligned}$$

3. The Local Projection Stabilized Method

3.1 *The method*

We introduce a local Stokes problem as a first step toward the full definition of the method. To this end, we use Araya *et al.* (2012) to set the following problem: Given $\boldsymbol{v}\in L^2(K)^d$, let $(\boldsymbol{u}_e^K(\boldsymbol{v}), p_e^K(\boldsymbol{v}))\in H_0^1(K)^d\times L_0^2(K)$ be the solution of the following (local) problem

$$\begin{aligned} -\nu\,\Delta\boldsymbol{u}_e^K(\boldsymbol{v}) + \nabla p_e^K(\boldsymbol{v}) = \boldsymbol{v}, \quad \nabla\cdot\boldsymbol{u}_e^K(\boldsymbol{v}) = 0 \quad \text{in} \quad K,\\ \boldsymbol{u}_e^K(\boldsymbol{v}) = \boldsymbol{0} \quad \text{on} \quad \partial K. \end{aligned} \tag{3.1}$$

The following result, proved in Lemma 3.2 of Barrenechea & Valentin (2010a), shows that the operator p_e^K is stable in the L^2 norm.

LEMMA 3.1 Let $\boldsymbol{v}\in L^2(K)^d$ and let $p_e^K(\boldsymbol{v})$ the solution of problem (3.1). Then, there exists $C>0$, independent of h_K, such that

$$\|p_e^K(\boldsymbol{v})\|_{0,K} \quad \leqslant \quad C\,h_K\,\|\boldsymbol{v}\|_{0,K}. \tag{3.2}$$

Now we are ready to present the LPS method for equation (2.1), which reads: *Find* $(\boldsymbol{u}_h,\tilde{p}_h)\in\boldsymbol{H}_h\times Q_h$ *such that*

$$\begin{aligned} &\nu(\nabla\boldsymbol{u}_h,\nabla\boldsymbol{v}_h) + ((\nabla\boldsymbol{u}_h)\boldsymbol{u}_h,\boldsymbol{v}_h) - (\tilde{p}_h,\nabla\cdot\boldsymbol{v}_h) + (\tilde{q}_h,\nabla\cdot\boldsymbol{u}_h)\\ &+\sum_{K\in\mathscr{T}_h}\frac{\alpha_K}{\nu}\left[(\chi_h(\tilde{p}_h),\chi_h(\tilde{q}_h))_K + (p_e^K((\nabla\boldsymbol{u}_h)\boldsymbol{u}_h), p_e^K((\nabla\boldsymbol{v}_h)\boldsymbol{u}_h))_K\right]\\ &+\sum_{K\in\mathscr{T}_h}\frac{\gamma_K}{\nu}\,(\chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{u}_h),\chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{v}_h))_K + \sum_{F\in\mathscr{E}_h}\tau_F\,([\![\tilde{p}_h]\!],[\![\tilde{q}_h]\!])_F = (\tilde{\boldsymbol{f}},\boldsymbol{v}_h), \end{aligned} \tag{3.3}$$

for all $(\boldsymbol{v}_h, \tilde{q}_h) \in \boldsymbol{H}_h \times Q_h$. The stabilization parameters are given by

$$\alpha_K := \frac{1}{\max\{1, Pe_K\}} \quad \text{and} \quad \gamma_K := \frac{1}{\max\left\{1, \frac{Pe_K}{24}\right\}},$$

with

$$Pe_K := \frac{|\boldsymbol{u}_h|_K h_K}{18\,\nu} \quad \text{with} \quad |\boldsymbol{u}_h|_K := \frac{\|\boldsymbol{u}_h\|_{0,K}}{|K|^{\frac{1}{2}}},$$

and

$$\tau_F := \begin{cases} \dfrac{h_F}{12\,\nu}, & \text{if } |\boldsymbol{u}_h|_F = 0, \\[2ex] \dfrac{1}{2\,|\boldsymbol{u}_h|_F} - \dfrac{1}{|\boldsymbol{u}_h|_F\,(1-\exp(Pe_F))}\Big(1 + \dfrac{1}{Pe_F}(1-\exp(Pe_F))\Big), & \text{otherwise.} \end{cases}$$

Here

$$Pe_F := \frac{|\boldsymbol{u}_h|_F\, h_F}{\nu} \quad \text{with} \quad |\boldsymbol{u}_h|_F := \frac{\|\boldsymbol{u}_h\|_{0,F}}{h_F^{1/2}}.$$

The numerical analysis of (3.3) will assume that the scaled version of the Navier-Stokes equations (2.15) holds. As such, and for sake of clarity, we adapt the LPS method (3.3) accordingly, which now takes the following form: *Find* $(\boldsymbol{u}_{h,\lambda}, p_{h,\lambda}) \in \boldsymbol{H}_h \times Q_h$ *such that for all* $(\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h$,

$$\begin{aligned} &(\nabla \boldsymbol{u}_{h,\lambda}, \nabla \boldsymbol{v}_h) + \lambda\,((\nabla \boldsymbol{u}_{h,\lambda})\boldsymbol{u}_{h,\lambda}, \boldsymbol{v}_h) - (p_{h,\lambda}, \nabla\cdot\boldsymbol{v}_h) + (q_h, \nabla\cdot\boldsymbol{u}_{h,\lambda}) \\ &+ \sum_{K\in\mathscr{T}_h} \alpha_K \left[(\chi_h(p_{h,\lambda}), \chi_h(q_h))_K + (p_e^K(\lambda\,(\nabla \boldsymbol{u}_{h,\lambda})\boldsymbol{u}_{h,\lambda}), p_e^K(\lambda\,(\nabla \boldsymbol{v}_h)\boldsymbol{u}_{h,\lambda}))_K\right] \\ &+ \sum_{K\in\mathscr{T}_h} \gamma_K \left(\lambda\,\chi_h(\boldsymbol{x}\nabla\cdot\boldsymbol{u}_{h,\lambda}), \lambda\,\chi_h(\boldsymbol{x}\nabla\cdot\boldsymbol{v}_h)\right)_K + \sum_{F\in\mathscr{E}_h} \tilde{\tau}_F\,\left([\![p_{h,\lambda}]\!], [\![q_h]\!]\right)_F = (\boldsymbol{f}, \boldsymbol{v}_h), \end{aligned} \tag{3.4}$$

where $\tilde{\tau}_F := \dfrac{\tau_F}{\lambda}$.

REMARK 3.1 The design of LPS method (3.3) is strongly inspired by the RELP method Araya *et al.* (2012), the former excluding the mixed pressure-velocity variational terms which are responsible for making the RELP method consistent. Another difference resides in the definition of the stabilized boundary terms which are now simpler (only pressure jumps are involved) than the ones proposed in the RELP version. Such simplifications may be seen as an example of the "minimal stabilization" concept developed in Brezzi & Fortin (2001), now used in the context of the incompressible Navier-Stokes equations. Indeed, we prove in the next sections that the stabilized terms are free to act differently and independently on the pressure and velocity variables without weakening the theoretical results obtained for original method (see Araya *et al.* (2012) for details).

3.2 *Existence and uniqueness*

Following the ideas presented in Tobiska & Verfürth (1996) and Araya *et al.* (2012), we start defining the operator $\mathscr{P} : \boldsymbol{H}_h \longrightarrow Q_h$ by

$$\sum_{K\in\mathscr{T}_h} \alpha_K \big(\chi_h(\mathscr{P}(\boldsymbol{u}_h)), \chi_h(q_h)\big)_K + \sum_{F\in\mathscr{E}_h} \tilde{\tau}_F\,\left([\![\mathscr{P}(\boldsymbol{u}_h)]\!], [\![q_h]\!]\right)_F = -(q_h, \nabla\cdot\boldsymbol{u}_h), \tag{3.5}$$

for all $q_h \in Q_h$. Note that the operator $\mathscr{P}$ is well defined due to Lax–Milgram's Lemma with the norm

$$\|q_h\|_* := \left\{ \sum_{K\in\mathscr{T}_h} \alpha_K \|\chi_h(q_h)\|^2_{0,K} + \sum_{F\in\mathscr{E}_h} \tilde{\tau}_F \, \|[\![q_h]\!]\|^2_{0,F} \right\}^{1/2}.$$

Also, we define the mapping $\mathscr{N} : \boldsymbol{H}_h \longrightarrow \boldsymbol{H}_h$ by

$$\begin{aligned}(\mathscr{N}(\boldsymbol{u}_h), \boldsymbol{v}_h) \quad := \quad & (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) + (\lambda(\nabla \boldsymbol{u}_h)\boldsymbol{u}_h, \boldsymbol{v}_h) - (\mathscr{P}(\boldsymbol{u}_h), \nabla\cdot\boldsymbol{v}_h) - (\boldsymbol{f}, \boldsymbol{v}_h) \\ & + \sum_{K\in\mathscr{T}_h} \alpha_K \left(\lambda p_e^K((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h), \lambda p_e^K((\nabla \boldsymbol{v}_h)\boldsymbol{u}_h)\right)_K \\ & + \sum_{K\in\mathscr{T}_h} \gamma_K \left(\lambda \chi_h(\boldsymbol{x}\nabla\cdot\boldsymbol{u}_h), \lambda\chi_h(\boldsymbol{x}\nabla\cdot\boldsymbol{v}_h)\right)_K,\end{aligned}$$

for all $\boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{H}_h$. The following result provides a characterization of the solution of (3.4) in terms of the mappings $\mathscr{P}$ and $\mathscr{N}$.

LEMMA 3.2 The pair $(\boldsymbol{u}_{h,\lambda}, p_{h,\lambda}) \in \boldsymbol{H}_h \times Q_h$ is a solution of problem (3.4) if and only if $\mathscr{N}(\boldsymbol{u}_{h,\lambda}) = \boldsymbol{0}$ and $p_{h,\lambda} = \mathscr{P}(\boldsymbol{u}_{h,\lambda})$.

Proof. For details see Lemma 3.5 in Araya *et al.* (2012). □

Next, we establish a sufficient condition assuring the existence of a discrete solution for (3.4).

THEOREM 3.1 There is a positive constant $\tilde{C}$, independent of λ and h, such that if

$$\lambda\, h^{1/2}\, \|\boldsymbol{f}\|_{-1,\Omega} \leqslant \tilde{C}, \tag{3.6}$$

then problem (3.4) admits at least one solution $(\boldsymbol{u}_{h,\lambda}, p_{h,\lambda}) \in \boldsymbol{H}_h \times Q_h$.

Proof. Let $\boldsymbol{u}_h \in \boldsymbol{H}_h$, with $|\boldsymbol{u}_h|_{1,\Omega} = R$, with R a positive constant to be fixed latter. In the sequel, we will use the following notation

$$x := \left\{ \sum_{K\in\mathscr{T}_h} \alpha_K \|p_e^K(\lambda(\nabla \boldsymbol{u}_h)\boldsymbol{u}_h)\|^2_{0,K} \right\}^{1/2}, \quad y := \left\{ \sum_{F\in\mathscr{E}_h} \tilde{\tau}_F \, \|[\![\mathscr{P}(\boldsymbol{u}_h)]\!]\|^2_{0,F} \right\}^{1/2},$$

$$z := \|\boldsymbol{f}\|_{-1,\Omega}, \quad w := \left\{ \sum_{K\in\mathscr{T}_h} \gamma_K \|\lambda\chi_h(\boldsymbol{x}\nabla\cdot\boldsymbol{u}_h)\|^2_{0,K} \right\}^{1/2}, \quad t := \left\{ \sum_{K\in\mathscr{T}_h} \alpha_K \|\chi_h(\mathscr{P}(\boldsymbol{u}_h))\|^2_{0,K} \right\}^{1/2}.$$

Taking $q_h = \mathscr{P}(\boldsymbol{u}_h)$ in (3.5) we get

$$-(\mathscr{P}(\boldsymbol{u}_h), \nabla\cdot\boldsymbol{u}_h) \quad = \quad \sum_{K\in\mathscr{T}_h} \alpha_K \|\chi_h(\mathscr{P}(\boldsymbol{u}_h))\|^2_{0,K} + \sum_{F\in\mathscr{E}_h} \tilde{\tau}_F \|[\![\mathscr{P}(\boldsymbol{u}_h)]\!]\|^2_{0,F}.$$

Using the above identity, Cauchy-Schwarz's inequality and (2.4), it holds

$$\begin{aligned}&(\mathscr{N}(\boldsymbol{u}_h), \boldsymbol{u}_h) = |\boldsymbol{u}_h|^2_{1,\Omega} + \lambda((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h, \boldsymbol{u}_h) - (\boldsymbol{f}, \boldsymbol{u}_h) + \sum_{K\in\mathscr{T}_h} \alpha_K \|\chi_h(\mathscr{P}(\boldsymbol{u}_h))\|^2_{0,K} \\ &+ \sum_{F\in\mathscr{E}_h} \tilde{\tau}_F \|[\![\mathscr{P}(\boldsymbol{u}_h)]\!]\|^2_{0,F} + \sum_{K\in\mathscr{T}_h} \alpha_K \|\lambda p_e^K((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h)\|^2_{0,K} + \sum_{K\in\mathscr{T}_h} \gamma_K \|\lambda\chi_h(\boldsymbol{x}\nabla\cdot\boldsymbol{u}_h)\|^2_{0,K} \\ &\geqslant R^2 + \lambda((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h, \boldsymbol{u}_h) - \|\boldsymbol{f}\|_{-1,\Omega} |\boldsymbol{u}_h|_{1,\Omega} + t^2 + y^2 + x^2 + w^2 \\ &\geqslant \frac{1}{2}R^2 + x^2 + y^2 + t^2 + w^2 - \frac{1}{2}z^2 - \frac{\lambda}{2}(\nabla\cdot\boldsymbol{u}_h, \boldsymbol{u}_h\cdot\boldsymbol{u}_h).\end{aligned} \tag{3.7}$$

Now, using Cauchy–Schwarz's inequality, Lemma 3.1, (3.5) with $q_h = \mathscr{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)$, the fact that $\alpha_K \leqslant 1$, (2.13) and the mesh regularity, we get

$$
\begin{aligned}
&|(\nabla\cdot\boldsymbol{u}_h,\boldsymbol{u}_h\cdot\boldsymbol{u}_h)| \leqslant |(\nabla\cdot\boldsymbol{u}_h,\boldsymbol{u}_h\cdot\boldsymbol{u}_h-\mathscr{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h))| + |(\nabla\cdot\boldsymbol{u}_h,\mathscr{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h))| \\
\leqslant\ & \sqrt{d}\,|\boldsymbol{u}_h|_{1,\Omega}\,\|\boldsymbol{u}_h\cdot\boldsymbol{u}_h-\mathscr{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)\|_{0,\Omega} + \left|\sum_{K\in\mathscr{T}_h}\alpha_K(\chi_h(\mathscr{P}(\boldsymbol{u}_h)),\chi_h(\mathscr{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)))_K\right| \\
& + \left|\sum_{F\in\mathscr{E}_h}\tilde{\tau}_F\,\|[\![\mathscr{P}(\boldsymbol{u}_h)]\!]\|_{0,F}\,\|[\![\mathscr{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)]\!]\|_{0,F}\right| \\
\leqslant\ & CR\left\{\sum_{K\in\mathscr{T}_h}h_K^2|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,K}^2\right\}^{1/2} + \sum_{K\in\mathscr{T}_h}\alpha_K\|\chi_h(\mathscr{P}(\boldsymbol{u}_h))\|_{0,K}\,\|\chi_h(\mathscr{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h))\|_{0,K} \\
& + \sum_{F\in\mathscr{E}_h}\tilde{\tau}_F\,\|[\![\mathscr{P}(\boldsymbol{u}_h)]\!]\|_{0,F}\,\|[\![\mathscr{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)]\!]\|_{0,F} \\
\leqslant\ & CR\left\{\sum_{K\in\mathscr{T}_h}h_K^2|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,K}^2\right\}^{1/2} + C\sum_{K\in\mathscr{T}_h}\alpha_K\|\chi_h(\mathscr{P}(\boldsymbol{u}_h))\|_{0,K}\,h_K|\mathscr{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)|_{1,K} \\
& + \sum_{F\in\mathscr{E}_h}\tilde{\tau}_F\,\|[\![\mathscr{P}(\boldsymbol{u}_h)]\!]\|_{0,F}\,\|[\![\mathscr{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)]\!]\|_{0,F} \\
\leqslant\ & CR\left\{\sum_{K\in\mathscr{T}_h}h_K^2|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,K}^2\right\}^{1/2} + Ct\left\{\sum_{K\in\mathscr{T}_h}\alpha_K h_K^2|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,\omega_K}^2\right\}^{1/2} + y\left\{\sum_{F\in\mathscr{E}_h}\tilde{\tau}_F\|[\![\mathscr{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)]\!]\|_{0,F}^2\right\}^{1/2} \\
\leqslant\ & C\{R+t+y\}\left\{\sum_{K\in\mathscr{T}_h}h_K^2|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,K}^2 + \sum_{F\in\mathscr{E}_h}\tilde{\tau}_F\|[\![\mathscr{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)-\boldsymbol{u}_h\cdot\boldsymbol{u}_h]\!]\|_{0,F}^2\right\}^{1/2}.
\end{aligned}
$$

Using that $\tilde{\tau}_F \leqslant Ch_F$ (see Lemma 2 in Barrenechea & Valentin (2010b)), (2.6), (2.13) and the mesh regularity, we have

$$
\begin{aligned}
\sum_{F\in\mathscr{E}_h}\tilde{\tau}_F\|[\![\mathscr{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)-\boldsymbol{u}_h\cdot\boldsymbol{u}_h]\!]\|_{0,F}^2 &\leqslant C\sum_{K\in\mathscr{T}_h}\|\mathscr{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)-\boldsymbol{u}_h\cdot\boldsymbol{u}_h\|_{0,K}^2 \\
&\leqslant C\sum_{K\in\mathscr{T}_h}h_K^2|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,K}^2,
\end{aligned}
$$

thus we arrive at

$$
|(\nabla\cdot\boldsymbol{u}_h,\boldsymbol{u}_h\cdot\boldsymbol{u}_h)| \leqslant C\{R+t+y\}\left\{\sum_{K\in\mathscr{T}_h}h_K^2|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,K}^2\right\}^{1/2}. \tag{3.8}
$$

Also, applying the local inverse inequality (2.5) with $l=m=0$, $p=\infty$ and $1\leqslant q\leqslant\infty$, we obtain that $\|v_h\|_{\infty,K}\leqslant Ch_K^{-\frac{d}{q}}\|v_h\|_{0,q,K}$. Next, using the Sobolev embedding $H^1(\Omega)\hookrightarrow L^q(\Omega)$, which is valid for all $2\leqslant q<\infty$ if $d=2$ and for $2\leqslant q\leqslant 6$, if $d=3$, we get

$$
\begin{aligned}
|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,K} &= \|\nabla(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)\|_{0,K} = 2\|\nabla(\boldsymbol{u}_h)\boldsymbol{u}_h\|_{0,K} \leqslant C|\boldsymbol{u}_h|_{1,K}\,\|\boldsymbol{u}_h\|_{\infty,K} \\
&\leqslant Ch_K^{-\frac{d}{q}}|\boldsymbol{u}_h|_{1,K}\,\|\boldsymbol{u}_h\|_{0,q,K} \leqslant Ch_K^{-\frac{d}{q}}|\boldsymbol{u}_h|_{1,K}\,\|\boldsymbol{u}_h\|_{0,q,\Omega} \leqslant Ch_K^{-\frac{d}{q}}|\boldsymbol{u}_h|_{1,K}\,|\boldsymbol{u}_h|_{1,\Omega}\,.
\end{aligned}
$$

Now, taking $q=4$, if $d=2$, or $q=6$, if $d=3$, we get

$$|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,K}\leqslant Ch_K^{-\frac{1}{2}}|\boldsymbol{u}_h|_{1,K}|\boldsymbol{u}_h|_{1,\Omega}$$

and then from (3.7) and (3.8), we arrive at

$$\begin{aligned}
&(\mathcal{N}(\boldsymbol{u}_h),\boldsymbol{u}_h)\geqslant\frac{1}{2}R^2+x^2+w^2+y^2+t^2-\frac{1}{2}z^2-\frac{\lambda}{2}(\nabla\cdot\boldsymbol{u}_h,\boldsymbol{u}_h\cdot\boldsymbol{u}_h)\\
\geqslant\;&\frac{1}{2}R^2+x^2+w^2+y^2+t^2-\frac{1}{2}z^2-C\frac{\lambda}{2}\{R+t+y\}\left\{\sum_{K\in\mathscr{T}_h}h_K|\boldsymbol{u}_h|_{1,K}^2\right\}^{1/2}|\boldsymbol{u}_h|_{1,\Omega}\\
\geqslant\;&\frac{1}{2}R^2+x^2+w^2+y^2+t^2-\frac{1}{2}z^2-Ch^{\frac{1}{2}}\lambda\{R+t+y\}R^2\\
\geqslant\;&\frac{1}{2}R^2+x^2+w^2+y^2+t^2-\frac{1}{2}z^2-Ch^{\frac{1}{2}}\lambda R^3-Ch^{\frac{1}{2}}\lambda\{t+y\}R^2\\
\geqslant\;&\frac{1}{2}R^2+x^2+w^2+y^2+t^2-\frac{1}{2}z^2-Ch^{\frac{1}{2}}\lambda R^3-\frac{1}{2}t^2-\frac{1}{2}y^2-\frac{3}{2}C^2h\lambda^2R^4\\
\geqslant\;&\frac{1}{2}R^2+x^2+\frac{1}{2}t^2+w^2+\frac{1}{2}y^2-z^2-Ch^{\frac{1}{2}}\lambda R^3-\frac{3}{2}C^2h\lambda^2R^4.
\end{aligned}$$

We define $R:=\dfrac{1}{6C\lambda h^{1/2}}$ and $\tilde{C}:=\dfrac{1}{12C}$, and note that (3.6) leads to

$$12C\lambda h^{1/2}z\leqslant 1\,.$$

Consequently, the definition of R leads to $z\leqslant\frac{R}{2}$, and then, we can gather the previous inequalities together to prove

$$\begin{aligned}
(\mathcal{N}(\boldsymbol{u}_h),\boldsymbol{u}_h)\;&\geqslant\;\left(\frac{1}{2}-\frac{1}{6}-\frac{3}{72}\right)R^2-z^2+x^2+\frac{1}{2}t^2+\frac{1}{2}y^2+w^2\\
&\geqslant\;\frac{1}{4}R^2-z^2+\frac{1}{2}y^2+x^2+\frac{1}{2}t^2+w^2\\
&\geqslant\;x^2+\frac{1}{2}y^2+\frac{1}{2}t^2+w^2\geqslant 0.
\end{aligned}$$

We conclude from Brouwer's fixed point Theorem (see Chapter IV, Corollary 1.1 in Girault & Raviart (1986)) that there exists a function $\boldsymbol{u}_{h,\lambda}\in\boldsymbol{H}_h$ such that $|\boldsymbol{u}_{h,\lambda}|_{1,\Omega}\leqslant R$ and $\mathcal{N}(\boldsymbol{u}_{h,\lambda})=\boldsymbol{0}$. $\square$

Now, we head to the question of uniqueness. To this end, we assume that λ is "small enough" such that $\alpha_K=\gamma_K=1$ on every element $K\in\mathscr{T}_h$. Also, on each edge (face) $F\in\mathscr{E}_h$ we take $\tilde{\tau}_F=\dfrac{h_F}{12}$ as both expressions are equivalent in the diffusion dominated case (see Lemma 2 in Barrenechea & Valentin (2010b) for details). Then, the method (3.4) simplifies to: *Find* $(\boldsymbol{u}_{h,\lambda},p_{h,\lambda})\in\boldsymbol{H}_h\times Q_h$ *such that*

$$\begin{aligned}
&(\nabla\boldsymbol{u}_{h,\lambda},\nabla\boldsymbol{v}_h)+\lambda\left((\nabla\boldsymbol{u}_{h,\lambda})\boldsymbol{u}_{h,\lambda},\boldsymbol{v}_h\right)-(p_{h,\lambda},\nabla\cdot\boldsymbol{v}_h)+(q_h,\nabla\cdot\boldsymbol{u}_{h,\lambda})\\
&+\sum_{K\in\mathscr{T}_h}(\chi_h(p_{h,\lambda}),\chi_h(q_h))_K+\sum_{K\in\mathscr{T}_h}(p_e^K(\lambda\,(\nabla\boldsymbol{u}_{h,\lambda})\boldsymbol{u}_{h,\lambda}),p_e^K(\lambda\,(\nabla\boldsymbol{v}_h)\boldsymbol{u}_{h,\lambda}))_K\\
&+\sum_{K\in\mathscr{T}_h}\left(\lambda\,\chi_h(\boldsymbol{x}\nabla\cdot\boldsymbol{u}_{h,\lambda}),\lambda\,\chi_h(\boldsymbol{x}\nabla\cdot\boldsymbol{v}_h)\right)_K+\sum_{F\in\mathscr{E}_h}\frac{h_F}{12}\left([\![p_{h,\lambda}]\!],[\![q_h]\!]\right)_F=(\boldsymbol{f},\boldsymbol{v}_h)\,,
\end{aligned}\tag{3.9}$$

for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h$. Next, method (3.9) is written as a fixed point problem through the mapping $G_h : \Lambda \times H^2(\mathscr{T}_h)^d \times H^1(\mathscr{T}_h) \longrightarrow \boldsymbol{H}_h \times Q_h$, where $G_h(\lambda, \boldsymbol{z}, t) = (\boldsymbol{w}_h, r_h)$ is such that

$$(\boldsymbol{w}_h, \boldsymbol{v}_h) + (r_h, q_h) = -(\boldsymbol{f} - \lambda(\nabla \boldsymbol{z})\boldsymbol{z}, \boldsymbol{v}_h) + \sum_{K \in \mathscr{T}_h} \left(p_e^K(\lambda\,(\nabla \boldsymbol{z})\boldsymbol{z}), p_e^K(\lambda\,(\nabla \boldsymbol{v}_h)\boldsymbol{z})\right)_K,$$

for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h$. As a result, problem (3.9) is analogous to the following problem

$$F_h(\lambda, \boldsymbol{u}_{h,\lambda}, p_{h,\lambda}) := (\boldsymbol{u}_{h,\lambda}, p_{h,\lambda}) + T_h G_h(\lambda, \boldsymbol{u}_{h,\lambda}, p_{h,\lambda}) = \boldsymbol{0}. \tag{3.10}$$

We are ready to present the uniqueness result.

THEOREM 3.2 Provide that λ is “sufficiently small”, in the sense that

$$\lambda\,(1+\lambda)\,(1+\lambda\,h)^2 < C,$$

where C is a given positive constant independent of h, the solution of problem (3.9) is unique.

Proof. First, observe that a solution of (3.9) is a fixed point of the operator $-T_h G_h$ using (3.10). Thereby, the proof reduces to show that the operator $-T_h G_h$ is a strict contraction in the ball $B := \{(\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h \,:\, |||(\boldsymbol{v}_h, q_h)||| \leqslant 1\}$, and use Banach's fixed point Theorem.

Let $(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h) \in B$. From Lemma 2.4 and the definition of operators T_h and G_h, it holds

$$\begin{aligned}
&|||T_h G_h(\lambda, \boldsymbol{u}_h, p_h) - T_h G_h(\lambda, \boldsymbol{v}_h, q_h)||| \leqslant C(1+\lambda\,h)^2 \sup_{|||(\boldsymbol{w}_h,t_h)|||\leqslant 1} (G_h(\lambda, \boldsymbol{u}_h, p_h) - G_h(\lambda, \boldsymbol{v}_h, q_h), (\boldsymbol{w}_h, t_h)) \\
&\leqslant C(1+\lambda\,h)^2 \sup_{|||(\boldsymbol{w}_h,t_h)|||\leqslant 1} \Big\{ \lambda((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h - (\nabla \boldsymbol{v}_h)\boldsymbol{v}_h, \boldsymbol{w}_h) + \sum_{K\in\mathscr{T}_h} \left(p_e^K(\lambda(\nabla \boldsymbol{u}_h)\boldsymbol{u}_h), p_e^K(\lambda(\nabla \boldsymbol{w}_h)\boldsymbol{u}_h)\right)_K \\
&- \sum_{K\in\mathscr{T}_h} \left(p_e^K(\lambda(\nabla \boldsymbol{v}_h)\boldsymbol{v}_h), p_e^K(\lambda(\nabla \boldsymbol{w}_h)\boldsymbol{v}_h)\right)_K \Big\} \\
&\leqslant C(1+\lambda\,h)^2 \sup_{|||(\boldsymbol{w}_h,t_h)|||\leqslant 1} \Big\{ \lambda((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h - (\nabla \boldsymbol{v}_h)\boldsymbol{v}_h, \boldsymbol{w}_h) + \sum_{K\in\mathscr{T}_h} \left(p_e^K(\lambda(\nabla \boldsymbol{u}_h)\boldsymbol{u}_h), p_e^K(\lambda(\nabla \boldsymbol{w}_h)(\boldsymbol{u}_h - \boldsymbol{v}_h))\right)_K \\
&+ \sum_{K\in\mathscr{T}_h} \left(p_e^K(\lambda((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h - (\nabla \boldsymbol{v}_h)\boldsymbol{v}_h), p_e^K(\lambda(\nabla \boldsymbol{w}_h)\boldsymbol{v}_h)\right)_K \Big\} \\
&= C(1+\lambda\,h)^2 \sup_{|||(\boldsymbol{w}_h,t_h)|||\leqslant 1} \Big\{ \mathrm{I} + \mathrm{II} + \mathrm{III} \Big\}.
\end{aligned} \tag{3.11}$$

We will estimate each term in the right-hand side of (3.11). First, using (2.3), and the following identity

$$(\nabla \boldsymbol{u}_h)\boldsymbol{u}_h - (\nabla \boldsymbol{v}_h)\boldsymbol{v}_h = \nabla(\boldsymbol{u}_h - \boldsymbol{v}_h)\boldsymbol{u}_h + \nabla \boldsymbol{v}_h(\boldsymbol{u}_h - \boldsymbol{v}_h)$$

and the definition of the norm $|||\cdot|||$, we get

$$\mathrm{I} \quad \leqslant \quad C\lambda\,|||(\boldsymbol{u}_h, p_h) - (\boldsymbol{v}_h, q_h)|||\,|||(\boldsymbol{w}_h, t_h)|||. \tag{3.12}$$

Next, from (2.5) with $m = 0$, $l = 1$ and $p = q = 4$, (3.2), Hölder, Cauchy–Schwarz and Poincaré inequalities, the inclusion $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and the definition of the norm $|||\cdot|||$, we obtain

$$
\begin{aligned}
\mathrm{II} &= \sum_{K\in\mathscr{T}_h} \left(p_e^K(\lambda(\nabla \boldsymbol{u}_h)\boldsymbol{u}_h), p_e^K(\lambda(\nabla \boldsymbol{w}_h)(\boldsymbol{u}_h-\boldsymbol{v}_h))\right)_K \\
&\leqslant C\sum_{K\in\mathscr{T}_h} \lambda^2 h_K^2 \|(\nabla \boldsymbol{u}_h)\boldsymbol{u}_h\|_{0,K}\|(\nabla \boldsymbol{w}_h)(\boldsymbol{u}_h-\boldsymbol{v}_h)\|_{0,K} \\
&\leqslant C\sum_{K\in\mathscr{T}_h} \lambda^2 h_K^2 \|\nabla \boldsymbol{u}_h\|_{0,4,K}\|\boldsymbol{u}_h\|_{0,4,K}\|\nabla \boldsymbol{w}_h\|_{0,4,K}\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_{0,4,K} \\
&\leqslant C\lambda^2 \sum_{K\in\mathscr{T}_h} h_K^2 h_K^{-1}\|\boldsymbol{u}_h\|_{0,4,K}\|\boldsymbol{u}_h\|_{0,4,K}\, h_K^{-1}\|\boldsymbol{w}_h\|_{0,4,K}\,\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_{0,4,K} \\
&\leqslant C\lambda^2\, |||(\boldsymbol{u}_h,p_h)|||^2 |||(\boldsymbol{u}_h,p_h)-(\boldsymbol{v}_h,q_h)||| |||(\boldsymbol{w}_h,t_h)||| \\
&\leqslant C\lambda^2\, |||(\boldsymbol{u}_h,p_h)-(\boldsymbol{v}_h,q_h)|||\, |||(\boldsymbol{w}_h,t_h)|||.
\end{aligned} \tag{3.13}
$$

Using similar arguments, we get

$$
\begin{aligned}
\mathrm{III} &= \sum_{K\in\mathscr{T}_h} \left(p_e^K(\lambda((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h-(\nabla \boldsymbol{v}_h)\boldsymbol{v}_h)), p_e^K(\lambda(\nabla \boldsymbol{w}_h)\boldsymbol{v}_h)\right)_K \\
&\leqslant C\lambda^2 \sum_{K\in\mathscr{T}_h} h_K^2 \|(\nabla \boldsymbol{u}_h)\boldsymbol{u}_h-(\nabla \boldsymbol{v}_h)\boldsymbol{v}_h\|_{0,K}\,\|\nabla \boldsymbol{w}_h\|_{0,4,K}\|\boldsymbol{v}_h\|_{0,4,K} \\
&\leqslant C\lambda^2 \left\{ |||(\boldsymbol{u}_h,p_h)||| + |||(\boldsymbol{v}_h,q_h)||| \right\} |||(\boldsymbol{u}_h,p_h)-(\boldsymbol{v}_h,q_h)||| |||(\boldsymbol{v}_h,q_h)||| |||(\boldsymbol{w}_h,t_h)||| \\
&\leqslant C\lambda^2 |||(\boldsymbol{u}_h,p_h)-(\boldsymbol{v}_h,q_h)||| |||(\boldsymbol{w}_h,t_h)|||.
\end{aligned} \tag{3.14}
$$

Now, replacing (3.12), (3.13) and (3.14), in (3.11), we see that

$$
|||T_hG_h(\lambda,\boldsymbol{u}_h,p_h)-T_hG_h(\lambda,\boldsymbol{v}_h,q_h)||| \leqslant C\lambda(1+\lambda)\,(1+\lambda h)^2\, |||(\boldsymbol{u}_h,p_h)-(\boldsymbol{v}_h,q_h)|||\,,
$$

and then the result follows if λ is such that $C\lambda\,(1+\lambda)\,(1+\lambda\, h)^2 < 1$. □

4. Error analysis

This section establishes a convergence result for the LPS method by extending the ideas proposed in Araya *et al.* (2012) for the RELP method. We place the method at the diffusion dominated regime for which the well–posedness is assured. The first result establishes that the differential operator $D_{\boldsymbol{u},p}F_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)$ is an isomorphism under appropriate conditions on h and λ. This result is key for the forthcoming error analysis.

LEMMA 4.1 Assume that a regular branch $\lambda \to (\boldsymbol{u}_\lambda, p_\lambda)$ of solutions of problem (2.17) exists on a given compact interval $\Lambda \subset \mathbb{R}$. Moreover, assume that $(\boldsymbol{u}_\lambda, p_\lambda)$ belongs to the space $H^2(\Omega)^d \times H^1(\Omega)$. Then, there exists a constant $h_0 > 0$ such that, for all $h \leqslant h_0$, the mapping $D_{\boldsymbol{u},p}F_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)$ is an isomorphism onto $\boldsymbol{H}_h \times Q_h$ and

$$
\|\{D_{\boldsymbol{u},p}F_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)\}^{-1}\|_{\mathscr{L}(\boldsymbol{H}_h\times Q_h)} \leqslant 8\,\|\{D_{\boldsymbol{u},p}F(\lambda,\boldsymbol{u}_\lambda,p_\lambda)\}^{-1}\|_{\mathscr{L}(\boldsymbol{H}\times Q)}. \tag{4.1}
$$

Proof. We start noting that, as T is a linear and continuous operator, from (2.3) and (2.9) we get

$$
\|D_{\boldsymbol{u},p}F(\lambda,\boldsymbol{u}_\lambda,p_\lambda) - D_{\boldsymbol{u},p}F(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)\|_{\mathscr{L}(\boldsymbol{H}\times Q)} \leqslant C\lambda\, hL,
$$

where

$$L := \sup_{\lambda\in\Lambda} \max\{\|\boldsymbol{f}\|_{0,\Omega}, \|\boldsymbol{u}_\lambda\|_{2,\Omega}, \|p_\lambda\|_{1,\Omega}\}. \tag{4.2}$$

Now, consider the linear operator $\mathscr{B} \in \mathscr{L}(\boldsymbol{H}\times Q)$ defined by

$$\mathscr{B} := \{D_{\boldsymbol{u},p}F(\lambda,\boldsymbol{u}_\lambda,p_\lambda)\}^{-1}\left[D_{\boldsymbol{u},p}F(\lambda,\boldsymbol{u}_\lambda,p_\lambda) - D_{\boldsymbol{u},p}F(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)\right].$$

Observe that $D_{\boldsymbol{u},p}F(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda) = D_{\boldsymbol{u},p}F(\lambda,\boldsymbol{u}_\lambda,p_\lambda)\,(\mathrm{I}-\mathscr{B})$ and since it holds

$$\lim_{h\to 0}\|\{D_{\boldsymbol{u},p}F(\lambda,\boldsymbol{u}_\lambda,p_\lambda)\}^{-1}\|_{\mathscr{L}(\boldsymbol{H}\times Q)}\,\|D_{\boldsymbol{u},p}F(\lambda,\boldsymbol{u}_\lambda,p_\lambda) - D_{\boldsymbol{u},p}F(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)\|_{\mathscr{L}(\boldsymbol{H}\times Q)} = 0,$$

then there exists $h_2 > 0$ such that $\|\mathscr{B}\|_{\mathscr{L}(\boldsymbol{H}\times Q)} \leqslant \frac{1}{2}$ for all $h \leqslant h_2$. As such, $\mathrm{I}-\mathscr{B}$ is an isomorphism onto $\boldsymbol{H}\times Q$ and

$$\|(\mathrm{I}-\mathscr{B})^{-1}\|_{\mathscr{L}(\boldsymbol{H}\times Q)} \leqslant \frac{1}{1-\|\mathscr{B}\|_{\mathscr{L}(\boldsymbol{H}\times Q)}} \leqslant 2. \tag{4.3}$$

We conclude that $D_{\boldsymbol{u},p}F(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)$ is also an isomorphism of $\boldsymbol{H}\times Q$ into itself for all $h\leqslant h_2$ and

$$\|\{D_{\boldsymbol{u},p}F(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)\}^{-1}\|_{\mathscr{L}(\boldsymbol{H}\times Q)} \leqslant 2\,\|\{D_{\boldsymbol{u},p}F(\lambda,\boldsymbol{u}_\lambda,p_\lambda)\}^{-1}\|_{\mathscr{L}(\boldsymbol{H}\times Q)}. \tag{4.4}$$

Now, notice that

$$D_{\boldsymbol{u},p}F_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda) = T_h\left[D_{\boldsymbol{u},p}G_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda) - D_{\boldsymbol{u},p}G(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)\right] + \mathscr{A}_1, \tag{4.5}$$

where the mapping $\mathscr{A}_1 := \mathrm{I} + T_h D_{\boldsymbol{u},p}G(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)$, belongs to $\mathscr{L}(\boldsymbol{H}\times Q)$, but also to $\mathscr{L}(\boldsymbol{H}_h\times Q_h)$. Then, we use Lemma 2.4, Hölder's inequality, (2.11), (2.12), and the inclusions $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$ and $H^1(\Omega)\hookrightarrow L^4(\Omega)$, to conclude that

$$\|\mathscr{A}_1 - D_{\boldsymbol{u},p}F(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)\|_{\mathscr{L}(\boldsymbol{H}\times Q)} \leqslant C\,(1+\lambda h)^2\,\lambda\,hL.$$

As a result, there exists $h_1 \leqslant h_2$ such that for all $h\leqslant h_1$, the mapping $\mathscr{A}_1$ is an isomorphism onto $\boldsymbol{H}\times Q$, and using (4.3), it holds

$$\|\mathscr{A}_1^{-1}\|_{\mathscr{L}(\boldsymbol{H}\times Q)} \leqslant 2\,\|\{D_{\boldsymbol{u},p}F(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)\}^{-1}\|_{\mathscr{L}(\boldsymbol{H}\times Q)}.$$

In addition, since $\mathscr{A}_1$ maps $\boldsymbol{H}_h\times Q_h$ onto itself and is a injective operator, then it is also an isomorphism onto $\boldsymbol{H}_h\times Q_h$ and

$$\|\mathscr{A}_1^{-1}\|_{\mathscr{L}(\boldsymbol{H}_h\times Q_h)} \leqslant \|\mathscr{A}_1^{-1}\|_{\mathscr{L}(\boldsymbol{H}\times Q)} \leqslant 2\,\|\{D_{\boldsymbol{u},p}F(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)\}^{-1}\|_{\mathscr{L}(\boldsymbol{H}\times Q)}. \tag{4.6}$$

Next, using Lemma 2.4 and (4.5), it holds

$$\begin{aligned}
&\|\mathscr{A}_1 - D_{\boldsymbol{u},p}F_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)\|_{\mathscr{L}(\boldsymbol{H}_h\times Q_h)}\\
=\;& \sup_{|||(\boldsymbol{v}_h,q_h)|||\leqslant 1} |||T_h(D_{\boldsymbol{u},p}G_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda) - D_{\boldsymbol{u},p}G(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda))[\boldsymbol{v}_h,q_h]|||\\
\leqslant\;& C\,(1+\lambda\,h)^2 \sup_{|||(\boldsymbol{v}_h,q_h)|||\leqslant 1} \|(D_{\boldsymbol{u},p}G_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda) - D_{\boldsymbol{u},p}G(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda))[\boldsymbol{v}_h,q_h]\|_{(\boldsymbol{H}_h\times Q_h)'}\\
\leqslant\;& C\,(1+\lambda\,h)^2 \sup_{|||(\boldsymbol{w}_h,t_h)|||\leqslant 1}\ \sup_{|||(\boldsymbol{v}_h,q_h)|||\leqslant 1} \Bigg\{ \sum_{K\in\mathscr{T}_h} \left(p_e^K(\lambda(\nabla\mathscr{I}_h\boldsymbol{u}_\lambda)\mathscr{I}_h\boldsymbol{u}_\lambda), p_e^K(\lambda(\nabla\boldsymbol{w}_h)\boldsymbol{v}_h)\right)_K\\
&+ \sum_{K\in\mathscr{T}_h} \left(p_e^K(\lambda(\nabla\boldsymbol{v}_h)\mathscr{I}_h\boldsymbol{u}_\lambda + \lambda(\nabla\mathscr{I}_h\boldsymbol{u}_\lambda)\boldsymbol{v}_h), p_e^K(\lambda(\nabla\boldsymbol{w}_h)\mathscr{I}_h\boldsymbol{u}_\lambda)\right)_K \Bigg\}\\
=\;& C\,(1+\lambda\,h)^2 \sup_{|||(\boldsymbol{w}_h,t_h)|||\leqslant 1}\ \sup_{|||(\boldsymbol{v}_h,q_h)|||\leqslant 1} \{\mathrm{I}+\mathrm{II}\}.
\end{aligned}$$

Let us estimate the right-hand side above. From (2.9), (2.5) with $l=1$, $m=0$ and $p=q=\infty$, (2.12) and the embedding $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$, we obtain

$$\begin{aligned} & \|(\nabla \boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda - (\nabla \mathscr{I}_h\boldsymbol{u}_\lambda)\mathscr{I}_h\boldsymbol{u}_\lambda\|_{0,K} \\ \leqslant\; & \|\nabla(\boldsymbol{u}_\lambda - \mathscr{I}_h\boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda\|_{0,K} + \|(\nabla \mathscr{I}_h\boldsymbol{u}_\lambda)(\boldsymbol{u}_\lambda - \mathscr{I}_h\boldsymbol{u}_\lambda)\|_{0,K} \\ \leqslant\; & |\boldsymbol{u}_\lambda - \mathscr{I}_h\boldsymbol{u}_\lambda|_{1,K}\|\boldsymbol{u}_\lambda\|_{\infty,K} + \|\nabla \mathscr{I}_h\boldsymbol{u}_\lambda\|_{\infty,K}\|\boldsymbol{u}_\lambda - \mathscr{I}_h\boldsymbol{u}_\lambda\|_{0,K} \\ \leqslant\; & Ch_K\|\boldsymbol{u}_\lambda\|_{2,K}\|\boldsymbol{u}_\lambda\|_{\infty,K} + Ch_K^{-1}\|\mathscr{I}_h\boldsymbol{u}_\lambda\|_{\infty,K}Ch_K^2\|\boldsymbol{u}_\lambda\|_{2,K} \\ \leqslant\; & Ch_K L\|\boldsymbol{u}_\lambda\|_{2,K}. \end{aligned} \tag{4.7}$$

Also, using the same arguments, we get

$$\|\lambda(\nabla \mathscr{I}_h\boldsymbol{u}_\lambda)\mathscr{I}_h\boldsymbol{u}_\lambda\|_{0,K} \leqslant CL\lambda\,\|\boldsymbol{u}_\lambda\|_{2,K}. \tag{4.8}$$

We estimate item I using Lemma 3.1, (4.8) and (2.5) with $l=1$, $m=0$, $p=q=4$, Poincaré and Hölder inequalities and the inclusion $H^1(\Omega)\hookrightarrow L^4(\Omega)$, as follows

$$\begin{aligned} \mathrm{I} &= \sum_{K\in\mathscr{T}_h}\left(p_e^K(\lambda(\nabla \mathscr{I}_h\boldsymbol{u}_\lambda)\mathscr{I}_h\boldsymbol{u}_\lambda), p_e^K(\lambda(\nabla \boldsymbol{w}_h)\boldsymbol{v}_h)\right)_K \\ &\leqslant C\sum_{K\in\mathscr{T}_h}\lambda\, h_K^2\|\lambda(\nabla \mathscr{I}_h\boldsymbol{u}_\lambda)\mathscr{I}_h\boldsymbol{u}_\lambda\|_{0,K}\|(\nabla \boldsymbol{w}_h)\boldsymbol{v}_h\|_{0,K} \\ &\leqslant C\sum_{K\in\mathscr{T}_h}\lambda^2\, h_K^2 L\|\boldsymbol{u}_\lambda\|_{2,K}\|\nabla \boldsymbol{w}_h\|_{0,4,K}\|\boldsymbol{v}_h\|_{0,4,K} \\ &\leqslant C\sum_{K\in\mathscr{T}_h}\lambda^2\, h_K^2 L\|\boldsymbol{u}_\lambda\|_{2,K}\, h_K^{-1}\|\boldsymbol{w}_h\|_{0,4,K}\|\boldsymbol{v}_h\|_{0,4,K} \\ &\leqslant C\lambda^2 L^2\, h\,|||(\boldsymbol{w}_h,t_h)|||\,|||(\boldsymbol{v}_h,q_h)|||. \end{aligned} \tag{4.9}$$

Concerning item II, we use again Lemma 3.1, the embeddings of $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$ and $H^1(\Omega)\hookrightarrow L^4(\Omega)$, (2.5) with $l=1$, $m=0$, $p=q=4$, (2.12), (2.10) and Poincaré and Hölder inequalities, to get

$$\begin{aligned} \mathrm{II} &= \sum_{K\in\mathscr{T}_h}\left(p_e^K(\lambda(\nabla \boldsymbol{v}_h)\mathscr{I}_h\boldsymbol{u}_\lambda + \lambda(\nabla \mathscr{I}_h\boldsymbol{u}_\lambda)\boldsymbol{v}_h), p_e^K(\lambda(\nabla \boldsymbol{w}_h)\mathscr{I}_h\boldsymbol{u}_\lambda)\right)_K \\ &\leqslant C\lambda^2\sum_{K\in\mathscr{T}_h}\|p_e^K((\nabla \boldsymbol{v}_h)\mathscr{I}_h\boldsymbol{u}_\lambda + (\nabla \mathscr{I}_h\boldsymbol{u}_\lambda)\boldsymbol{v}_h)\|_{0,K}\|p_e^K((\nabla \boldsymbol{w}_h)\mathscr{I}_h\boldsymbol{u}_\lambda)\|_{0,K} \\ &\leqslant C\lambda^2\sum_{K\in\mathscr{T}_h}h_K^2\|(\nabla \boldsymbol{v}_h)\mathscr{I}_h\boldsymbol{u}_\lambda + (\nabla \mathscr{I}_h\boldsymbol{u}_\lambda)\boldsymbol{v}_h\|_{0,K}\|(\nabla \boldsymbol{w}_h)\mathscr{I}_h\boldsymbol{u}_\lambda\|_{0,K} \\ &\leqslant C\lambda^2\sum_{K\in\mathscr{T}_h}h_K^2\left\{\|\nabla \boldsymbol{v}_h\|_{0,K}\|\mathscr{I}_h\boldsymbol{u}_\lambda\|_{\infty,K} + \|\nabla \mathscr{I}_h\boldsymbol{u}_\lambda\|_{0,4,K}\|\boldsymbol{v}_h\|_{0,4,K}\right\}\|\nabla \boldsymbol{w}_h\|_{0,K}\|\mathscr{I}_h\boldsymbol{u}_\lambda\|_{\infty,K} \\ &\leqslant C\lambda^2\sum_{K\in\mathscr{T}_h}h_K^2\left\{\|\nabla \boldsymbol{v}_h\|_{0,K}\|\boldsymbol{u}_\lambda\|_{\infty,K} + h_K^{-1}\|\mathscr{I}_h\boldsymbol{u}_\lambda\|_{0,4,K}\|\boldsymbol{v}_h\|_{0,4,K}\right\}\|\nabla \boldsymbol{w}_h\|_{0,K}\|\boldsymbol{u}_\lambda\|_{\infty,K} \\ &\leqslant C\lambda^2 L^2 h(1+h)\;|||(\boldsymbol{v}_h,q_h)|||\,|||(\boldsymbol{w}_h,t_h)|||. \end{aligned} \tag{4.10}$$

Therefore, from (4.9) and (4.10) it follows that

$$\|\mathscr{A}_1 - D_{\boldsymbol{u},p}F_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)\|_{\mathscr{L}(\boldsymbol{H}_h\times Q_h)} \leqslant C\,h\,(1+\lambda\,h)^2\,(1+h)\,\lambda^2 L^2,$$

and it results that there exists $h_0 \leqslant h_1$ such that for all $h \leqslant h_0$ the mapping $D_{\boldsymbol{u},p}F_h(\lambda, \mathscr{I}_h\boldsymbol{u}_\lambda, \mathscr{J}_h p_\lambda)$ is an isomorphism of $\boldsymbol{H}_h \times Q_h$ into itself and

$$\|\{D_{\boldsymbol{u},p}F_h(\lambda, \mathscr{I}_h\boldsymbol{u}_\lambda, \mathscr{J}_h p_\lambda)\}^{-1}\|_{\mathscr{L}(\boldsymbol{H}_h\times Q_h)} \leqslant 2\,\|\mathscr{A}_1^{-1}\|_{\mathscr{L}(\boldsymbol{H}\times Q)}. \tag{4.11}$$

Finally, from (4.4), (4.6) and (4.11) we obtain (4.1). □

Since the LPS method lacks of consistency, it is important to estimate such an error in order to prove that it does not deteriorate the convergence rates. This is the subject of the next theorem.

THEOREM 4.1 Let $(\boldsymbol{u}_\lambda, p_\lambda)$ be a solution of (2.15) and assume that $(\boldsymbol{u}_\lambda, p_\lambda) \in H^2(\Omega)^d \times H^1(\Omega)$. Let F_h be the mapping defined in (3.10). Then, we have the following estimate

$$|||F_h(\lambda, \boldsymbol{u}_\lambda, p_\lambda)||| \leqslant C\,hL\,(1 + h\lambda^2 L^2),$$

where L is given in (4.2).

Proof. From (2.7), (2.5) and the continuous embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we get

$$\begin{aligned}
|||F_h(\lambda, \boldsymbol{u}_\lambda, p_\lambda)||| &= \sup_{|||(\boldsymbol{v}_h,q_h)|||\leqslant 1} (F_h(\lambda, \boldsymbol{u}_\lambda, p_\lambda), (\boldsymbol{v}_h, q_h))\\
&= \sup_{|||(\boldsymbol{v}_h,q_h)|||\leqslant 1} \left\{ \sum_{K\in\mathscr{T}_h} \Big[(\chi_h(p_\lambda), \chi_h(q_h))_K + \big(p_e^K(\lambda(\nabla\boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda), p_e^K(\lambda(\nabla\boldsymbol{v}_h)\boldsymbol{u}_\lambda)\big)_K \Big] \right\}\\
&\leqslant C \sup_{|||(\boldsymbol{v}_h,q_h)|||\leqslant 1} \left\{ \sum_{K\in\mathscr{T}_h} h_K^2 \Big[|p_\lambda|_{1,K} |q_h|_{1,K} + \lambda^2 \|(\nabla\boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda\|_{0,K} \|(\nabla\boldsymbol{v}_h)\boldsymbol{u}_\lambda\|_{0,K} \Big] \right\}\\
&\leqslant C \sup_{|||(\boldsymbol{v}_h,q_h)|||\leqslant 1} \left\{ \sum_{K\in\mathscr{T}_h} h_K^2 \Big[|p_\lambda|_{1,K}\, h_K^{-1} \|q_h\|_{0,K} + \lambda^2 \|\boldsymbol{u}_\lambda\|_{\infty,K}^2 |\boldsymbol{u}_\lambda|_{1,K} |\boldsymbol{v}_h|_{1,K} \Big] \right\}\\
&\leqslant C \sup_{|||(\boldsymbol{v}_h,q_h)|||\leqslant 1} \left\{ h \Big[|p_\lambda|_{1,\Omega} \|q_h\|_{0,\Omega} + \lambda^2 h \|\boldsymbol{u}_\lambda\|_{\infty,\Omega}^2 |\boldsymbol{u}_\lambda|_{1,\Omega} |\boldsymbol{v}_h|_{1,\Omega} \Big] \right\}\\
&\leqslant C hL(1 + h\lambda^2 L^2),
\end{aligned}$$

and the result follows. □

The next result states further properties of the mapping F_h and its derivative which will be used to prove error estimates.

LEMMA 4.2 Assume the hypothesis of Lemma 4.1 hold. Therefore, there exists a constant C, which is independent of h and λ, such that

$$|||F_h(\lambda, \mathscr{I}_h\boldsymbol{u}_\lambda, \mathscr{J}_h p_\lambda)||| \leqslant C\,hL\left\{2 + \lambda L(1+\lambda h)^2(1+\lambda hL) + h\lambda^2 L^2\right\}. \tag{4.12}$$

Furthermore, for each $\rho > 0$ and for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h$ such that $(\boldsymbol{v}_h, q_h)$ belongs to the ball centered at $(\mathscr{I}_h\boldsymbol{u}_\lambda, \mathscr{J}_h p_\lambda)$ with radius ρ, there exists a constant $C > 0$, independent of h and λ but depending on ρ, such that

$$\begin{aligned}
&\|D_{\boldsymbol{u},p}F_h(\lambda, \mathscr{I}_h\boldsymbol{u}_\lambda, \mathscr{J}_h p_\lambda) - D_{\boldsymbol{u},p}F_h(\lambda, \boldsymbol{v}_h, q_h)\|_{\mathscr{L}(\boldsymbol{H}_h\times Q_h)}\\
&\qquad \leqslant C\lambda\,(1+\lambda h)^2\,(1+\lambda L)\,|||(\mathscr{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h, \mathscr{J}_h p_\lambda - q_h)|||.
\end{aligned} \tag{4.13}$$

Proof. From the linearity of the operator T_h we obtain

$$
\begin{aligned}
&|||F_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)|||\\
&\quad= |||F_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)-F_h(\lambda,\boldsymbol{u}_\lambda,p_\lambda)|||+|||F_h(\lambda,\boldsymbol{u}_\lambda,p_\lambda)|||\\
&\quad\leqslant |||(\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda-p_\lambda)|||+|||T_h(G_h(\lambda,\boldsymbol{u}_\lambda,p_\lambda)-G_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda))|||+|||F_h(\lambda,\boldsymbol{u}_\lambda,p_\lambda)|||\\
&\quad= S_1+S_2+S_3.
\end{aligned}
\tag{4.14}
$$

To estimate S_1, we use (2.9) and (2.13) and we get

$$
S_1 \leqslant C\,h\,L. \tag{4.15}
$$

Next, using the continuity of T_h (see Lemma 2.4) and the definition of the dual norm, it holds

$$
\begin{aligned}
S_2 &= |||T_h(G_h(\lambda,\boldsymbol{u}_\lambda,p_\lambda)-G_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda))|||\\
&\leqslant C(1+\lambda\,h)^2 \sup_{|||(\boldsymbol{v}_h,q_h)|||\leqslant 1}(G_h(\lambda,\boldsymbol{u}_\lambda,p_\lambda)-G_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda),(\boldsymbol{v}_h,q_h))\\
&\leqslant C(1+\lambda\,h)^2 \sup_{|||(\boldsymbol{v}_h,q_h)|||\leqslant 1}\left\{\lambda((\nabla\boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda-(\nabla\mathscr{I}_h\boldsymbol{u}_\lambda)\mathscr{I}_h\boldsymbol{u}_\lambda,\boldsymbol{v}_h)_\Omega\right.\\
&\qquad+\sum_{K\in\mathscr{T}_h}\left(p_e^K(\lambda(\nabla\boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda),p_e^K(\lambda(\nabla\boldsymbol{v}_h)\boldsymbol{u}_\lambda)\right)_K\\
&\qquad\left.-\sum_{K\in\mathscr{T}_h}\left(p_e^K(\lambda(\nabla\mathscr{I}_h\boldsymbol{u}_\lambda)\mathscr{I}_h\boldsymbol{u}_\lambda),p_e^K(\lambda(\nabla\boldsymbol{v}_h)\mathscr{I}_h\boldsymbol{u}_\lambda)\right)_K\right\}\\
&\leqslant C(1+\lambda\,h)^2 \sup_{|||(\boldsymbol{v}_h,q_h)|||\leqslant 1}\left\{\mathrm{I}+\mathrm{II}+\mathrm{III}\right\}.
\end{aligned}
$$

As for the first term on the right-hand side above, we use (4.7), (4.8) and Cauchy–Schwarz and Poincaré inequalities, to get

$$
\mathrm{I} \leqslant C\,h\,\lambda\,L^2\,|||(\boldsymbol{v}_h,q_h)|||. \tag{4.16}
$$

We estimate item II, through Cauchy–Schwarz's inequality, (3.2) and the continuous embedding $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$ as follows

$$
\begin{aligned}
\mathrm{II} &= \sum_{K\in\mathscr{T}_h}\left(p_e^K(\lambda(\nabla\boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda),p_e^K(\lambda(\nabla\boldsymbol{v}_h)\boldsymbol{u}_\lambda)\right)_K\\
&\leqslant C\sum_{K\in\mathscr{T}_h} h_K^2\,\|\lambda(\nabla\boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda\|_{0,K}\|\lambda(\nabla\boldsymbol{v}_h)\boldsymbol{u}_\lambda\|_{0,K}\\
&\leqslant C\sum_{K\in\mathscr{T}_h}\lambda^2 h_K^2\,\|\boldsymbol{u}_\lambda\|_{\infty,K}^2|\boldsymbol{u}_\lambda|_{1,K}|\boldsymbol{v}_h|_{1,K}\\
&\leqslant C\lambda^2 h^2 L^3\,|||(\boldsymbol{v}_h,q_h)|||.
\end{aligned}
\tag{4.17}
$$

Item III is bounded using Lemma 3.1, (4.8) and the continuous embedding $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$, and (4.8) as follows

$$\begin{aligned}
\mathrm{III} &= \sum_{K\in\mathscr{T}_h}\big(p_e^K(\lambda(\nabla\mathscr{I}_h\boldsymbol{u}_\lambda)\mathscr{I}_h\boldsymbol{u}_\lambda), p_e^K(\lambda(\nabla\boldsymbol{v}_h)\mathscr{I}_h\boldsymbol{u}_\lambda)\big)_K \\
&\leqslant C\sum_{K\in\mathscr{T}_h} h_K^2\,\|\lambda(\nabla\mathscr{I}_h\boldsymbol{u}_\lambda)\mathscr{I}_h\boldsymbol{u}_\lambda\|_{0,K}\|\lambda(\nabla\boldsymbol{v}_h)\mathscr{I}_h\boldsymbol{u}_\lambda\|_{0,K} \\
&\leqslant C\sum_{K\in\mathscr{T}_h}\lambda^2 h_K^2 L\,\|\boldsymbol{u}_\lambda\|_{2,K}|\boldsymbol{v}_h|_{1,K}\|\mathscr{I}_h\boldsymbol{u}_\lambda\|_{\infty,K} \\
&\leqslant C\sum_{K\in\mathscr{T}_h}\lambda^2 h_K^2 L\,\|\boldsymbol{u}_\lambda\|_{2,K}|\boldsymbol{v}_h|_{1,K}\|\boldsymbol{u}_\lambda\|_{\infty,K} \\
&\leqslant C\lambda^2 h^2 L^3\,|||(\boldsymbol{v}_h,q_h)|||\,.
\end{aligned}\tag{4.18}$$

Now, gathering (4.16)–(4.18) together, we obtain the following estimate

$$S_2 \leqslant C\,h\,(1+\lambda\,h)^2\,(1+\lambda\,hL)\lambda\,L^2\,, \tag{4.19}$$

and from (4.14) and (4.15), (4.19), and Theorem 4.1 result (4.12) follows. Estimate (4.13) is addressed next. Let $(\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h),(\boldsymbol{w}_h,r_h)\in\boldsymbol{H}_h\times Q_h$ with $|||(\boldsymbol{w}_h,r_h)|||=1$. From the stability of the discrete Stokes operator in Lemma 2.4, we get

$$\begin{aligned}
&|||D_{\boldsymbol{u},p}F_h(\lambda,\boldsymbol{v}_h,q_h)(\boldsymbol{w}_h,r_h)-D_{\boldsymbol{u},p}F_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_hp_\lambda)(\boldsymbol{w}_h,r_h)||| \\
=\;& |||T_h(D_{\boldsymbol{u},p}G_h(\lambda,\boldsymbol{v}_h,q_h)(\boldsymbol{w}_h,r_h)-D_{\boldsymbol{u},p}G_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_hp_\lambda)(\boldsymbol{w}_h,r_h))||| \\
\leqslant\;& C\,(1+\lambda\,h)^2\sup_{|||(\boldsymbol{z}_h,s_h)|||\leqslant 1}(D_{\boldsymbol{u},p}G_h(\lambda,\boldsymbol{v}_h,q_h)(\boldsymbol{w}_h,r_h)-D_{\boldsymbol{u},p}G_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_hp_\lambda)(\boldsymbol{w}_h,r_h),(\boldsymbol{z}_h,s_h)) \\
\leqslant\;& C\,(1+\lambda\,h)^2\sup_{|||(\boldsymbol{z}_h,s_h)|||\leqslant 1}\Big\{(\lambda\nabla(\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h)\boldsymbol{w}_h-\lambda(\nabla\boldsymbol{w}_h)(\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h),\boldsymbol{z}_h) \\
&+\sum_{K\in\mathscr{T}_h}\big(p_e^K(\lambda(\nabla\boldsymbol{v}_h)\boldsymbol{v}_h-\lambda(\nabla\mathscr{I}_h\boldsymbol{u}_\lambda)\mathscr{I}_h\boldsymbol{u}_\lambda),p_e^K(\lambda(\nabla\boldsymbol{z}_h)\boldsymbol{w}_h)\big)_K \\
&-\sum_{K\in\mathscr{T}_h}\big(p_e^K(\lambda(\nabla\mathscr{I}_h\boldsymbol{u}_\lambda)\boldsymbol{w}_h+\lambda(\nabla\boldsymbol{w}_h)\mathscr{I}_h\boldsymbol{u}_\lambda),p_e^K(\lambda(\nabla\boldsymbol{z}_h)\mathscr{I}_h\boldsymbol{u}_\lambda)\big)_K \\
&+\sum_{K\in\mathscr{T}_h}\big(p_e^K(\lambda(\nabla\boldsymbol{v}_h)\boldsymbol{w}_h+\lambda(\nabla\boldsymbol{w}_h)\boldsymbol{v}_h),p_e^K(\lambda(\nabla\boldsymbol{z}_h)\boldsymbol{v}_h)\big)_K \\
=\;& C\,(1+\lambda\,h)^2\sup_{|||(\boldsymbol{z}_h,s_h)|||\leqslant 1}\Big\{\mathrm{IV}+\mathrm{V}+\mathrm{VI}+\mathrm{VII}\Big\}.
\end{aligned}$$

We bound IV using (2.3) as follows

$$\mathrm{IV}\leqslant 2\,\alpha\,\lambda\,|||(\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h,\mathscr{J}_hp_\lambda-q_h)|||\;|||(\boldsymbol{w}_h,r_h)|||\,|||(\boldsymbol{z}_h,s_h)|||\,. \tag{4.20}$$

We address item V now. To this end, we use Lemma 3.1, (2.5) with $l=1$, $m=0$ and $p=q=4$, the embedding $H^1(\Omega)\hookrightarrow L^4(\Omega)$, Hölder, Cauchy–Schwarz and Poincaré inequalities and the definition of the norm $|||\cdot|||$, to obtain

$$
\begin{aligned}
\mathrm{V} &= -\sum_{K\in\mathscr{T}_h}\lambda\left(p_e^K((\nabla\mathscr{I}_h\boldsymbol{u}_\lambda)\mathscr{I}_h\boldsymbol{u}_\lambda-(\nabla\boldsymbol{v}_h)\boldsymbol{v}_h),p_e^K(\lambda(\nabla\boldsymbol{z}_h)\boldsymbol{w}_h)\right)_K\\
&\leqslant C\lambda^2\sum_{K\in\mathscr{T}_h}h_K^2\|(\nabla\mathscr{I}_h\boldsymbol{u}_\lambda)\mathscr{I}_h\boldsymbol{u}_\lambda-(\nabla\boldsymbol{v}_h)\boldsymbol{v}_h\|_{0,K}\|(\nabla\boldsymbol{z}_h)\boldsymbol{w}_h\|_{0,K}\\
&\leqslant C\lambda^2\sum_{K\in\mathscr{T}_h}h_K^2\|(\nabla\mathscr{I}_h\boldsymbol{u}_\lambda)\mathscr{I}_h\boldsymbol{u}_\lambda-(\nabla\boldsymbol{v}_h)\boldsymbol{v}_h\|_{0,K}|\boldsymbol{z}_h|_{1,4,K}\|\boldsymbol{w}_h\|_{0,4,K}\\
&\leqslant C\lambda^2\sum_{K\in\mathscr{T}_h}h_K^2\{\|(\nabla\mathscr{I}_h\boldsymbol{u}_\lambda)(\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h)\|_{0,K}+\|(\nabla(\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h))\boldsymbol{v}_h\|_{0,K}\}h_K^{-1}\|\boldsymbol{z}_h\|_{0,4,K}\|\boldsymbol{w}_h\|_{0,4,K}\\
&\leqslant C\lambda^2\sum_{K\in\mathscr{T}_h}h_K^2\{\|\nabla\mathscr{I}_h\boldsymbol{u}_\lambda\|_{0,4,K}\|\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h\|_{0,4,K}+\|\nabla(\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h)\|_{0,4,K}\|\boldsymbol{v}_h\|_{0,4,K}\}h_K^{-1}\|\boldsymbol{z}_h\|_{0,4,K}\|\boldsymbol{w}_h\|_{0,4,K}\\
&\leqslant C\lambda^2\sum_{K\in\mathscr{T}_h}h_K^2\left\{h_K^{-1}\|\mathscr{I}_h\boldsymbol{u}_\lambda\|_{0,4,K}\|\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h\|_{0,4,K}+h_K^{-1}\|\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h\|_{0,4,K}\|\boldsymbol{v}_h\|_{0,4,K}\right\}h_K^{-1}\|\boldsymbol{z}_h\|_{0,4,K}\|\boldsymbol{w}_h\|_{0,4,K}\\
&\leqslant C\lambda^2\left\{\|\mathscr{I}_h\boldsymbol{u}_\lambda\|_{0,4,\Omega}+\|\boldsymbol{v}_h\|_{0,4,\Omega}\right\}\|\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h\|_{0,4,\Omega}\|\boldsymbol{z}_h\|_{0,4,\Omega}\|\boldsymbol{w}_h\|_{0,4,\Omega}\\
&\leqslant C\lambda^2\left\{\|\mathscr{I}_h\boldsymbol{u}_\lambda\|_{1,\Omega}+\|\boldsymbol{v}_h\|_{1,\Omega}\right\}\|\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h\|_{1,\Omega}\|\boldsymbol{z}_h\|_{1,\Omega}\|\boldsymbol{w}_h\|_{1,\Omega}\\
&\leqslant C\lambda^2\{L+\rho\}|||(\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h,\mathscr{J}_hp_\lambda-q_h)|||\,|||(\boldsymbol{w}_h,r_h)|||\,|||(\boldsymbol{z}_h,s_h)|||.
\end{aligned}\tag{4.21}
$$

We follow closely the same arguments to estimate items VI and VII, and we get

$$
\begin{aligned}
\mathrm{VI} &= -\sum_{K\in\mathscr{T}_h}\lambda\left(p_e^K((\nabla(\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h))\boldsymbol{w}_h+(\nabla\boldsymbol{w}_h)(\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h)),p_e^K(\lambda(\nabla\boldsymbol{z}_h)\mathscr{I}_h\boldsymbol{u}_\lambda)\right)_K\\
&\leqslant C\lambda^2\sum_{K\in\mathscr{T}_h}h_K^2\|(\nabla(\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h))\boldsymbol{w}_h+(\nabla\boldsymbol{w}_h)(\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h)\|_{0,K}\|(\nabla\boldsymbol{z}_h)\mathscr{I}_h\boldsymbol{u}_\lambda\|_{0,K}\\
&\leqslant C\lambda^2\sum_{K\in\mathscr{T}_h}h_K^2\{\|\nabla(\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h)\|_{0,4,K}\|\boldsymbol{w}_h\|_{0,4,K}+\|\nabla\boldsymbol{w}_h\|_{0,4,K}\|\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h\|_{0,4,K}\}\|\nabla\boldsymbol{z}_h\|_{0,4,K}\|\mathscr{I}_h\boldsymbol{u}_\lambda\|_{0,4,K}\\
&\leqslant C\lambda^2L|||(\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h,\mathscr{J}_hp_\lambda-q_h)|||\,|||(\boldsymbol{w}_h,r_h)|||\,|||(\boldsymbol{z}_h,s_h)|||,
\end{aligned}\tag{4.22}
$$

and

$$
\begin{aligned}
\mathrm{VII} &= -\sum_{K\in\mathscr{T}_h}\lambda\left(p_e^K((\nabla\boldsymbol{v}_h)\boldsymbol{w}_h+(\nabla\boldsymbol{w}_h)\boldsymbol{v}_h),p_e^K(\lambda(\nabla\boldsymbol{z}_h)(\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h))\right)_K\\
&\leqslant C\lambda^2\sum_{K\in\mathscr{T}_h}h_K^2\|(\nabla\boldsymbol{v}_h)\boldsymbol{w}_h+(\nabla\boldsymbol{w}_h)\boldsymbol{v}_h\|_{0,K}\|(\nabla\boldsymbol{z}_h)(\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h)\|_{0,K}\\
&\leqslant C\lambda^2\sum_{K\in\mathscr{T}_h}h_K^2\{\|\nabla\boldsymbol{v}_h\|_{0,4,K}\|\boldsymbol{w}_h\|_{0,4,K}+\|\nabla\boldsymbol{w}_h\|_{0,4,K}\|\boldsymbol{v}_h\|_{0,4,K}\}\|\boldsymbol{z}_h\|_{0,4,K}\|\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h\|_{0,4,K}\\
&\leqslant C\lambda^2\rho\,|||(\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h,\mathscr{J}_hp_\lambda-q_h)|||\,|||(\boldsymbol{w}_h,r_h)|||\,|||(\boldsymbol{z}_h,s_h)|||.
\end{aligned}\tag{4.23}
$$

Estimate (4.13) results from (4.20)–(4.23). □

We are now ready to prove the existence and uniqueness of a branch of discrete solutions as well as the error estimate.

THEOREM 4.2 Assume the hypothesis of Lemma 4.1 hold. Therefore, there exists a positive constant $h_0(\Lambda)$, such that for all h with $0<h\leqslant h_0$, a unique branch $\lambda\to(\boldsymbol{u}_{h,\lambda},p_{h,\lambda})$ of solutions of problem

(3.10) exists in a neighborhood of $(\boldsymbol{u}_\lambda, p_\lambda)$. Moreover, the following estimate holds

$$\sup_{\lambda\in\Lambda}\left\{|\boldsymbol{u}_\lambda-\boldsymbol{u}_{h,\lambda}|_{1,\Omega}^2+\|p_\lambda-p_{h,\lambda}\|_{0,\Omega}^2\right\}^{1/2}\leqslant Ch,$$

where $C=C(L,\Lambda)>0$ does not depend on h.

Proof. As a result of Lemma 4.1, the differential operator $D_{\boldsymbol{u},p}F_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)$ is an isomorphism of $\boldsymbol{H}_h\times Q_h$ onto itself for each $\lambda\in\Lambda$, provided that $h\sup_{\lambda\in\Lambda}\lambda$ is sufficiently small. Now, let $\Phi:\boldsymbol{H}_h\times Q_h\to\boldsymbol{H}_h\times Q_h$ be the application defined by

$$\Phi(\boldsymbol{u}_h,p_h):=(\boldsymbol{u}_h,p_h)+D_{\boldsymbol{u},p}F_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)^{-1}F_h(\lambda,\boldsymbol{u}_h,p_h)\qquad\forall(\boldsymbol{u}_h,p_h)\in\boldsymbol{H}_h\times Q_h\,.$$

Observe that (3.10) may be rewritten as a fixed point problem with respect to the operator Φ. Next, let $(\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h)\in B((\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda),\rho)$, where ρ will be fixed in the sequel. From Lemma 4.2 we get

$$\begin{aligned}
&|||\Phi(\boldsymbol{u}_h,p_h)-\Phi(\boldsymbol{v}_h,q_h)|||\leqslant\|\{D_{\boldsymbol{u},p}F_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)\}^{-1}\|_{\mathscr{L}(\boldsymbol{H}_h\times Q_h)}\\
&\quad|||F_h(\lambda,\boldsymbol{u}_h,p_h)-F_h(\lambda,\boldsymbol{v}_h,q_h)-D_{\boldsymbol{u},p}F_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)[(\boldsymbol{u}_h,p_h)-(\boldsymbol{v}_h,q_h)]|||\\
\leqslant\;&\|\{D_{\boldsymbol{u},p}F_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)\}^{-1}\|_{\mathscr{L}(\boldsymbol{H}_h\times Q_h)}\left|\left|\left|\int_0^1\left\{D_{\boldsymbol{u},p}F_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)\right.\right.\right.\right.\\
&\quad\left.\left.\left.\left.-D_{\boldsymbol{u},p}F_h(\lambda,\boldsymbol{u}_h+\theta(\boldsymbol{v}_h-\boldsymbol{u}_h),p_h+\theta(q_h-p_h))\right\}\left[(\boldsymbol{u}_h,p_h)-(\boldsymbol{v}_h,q_h)\right]d\theta\right|\right|\right|\\
\leqslant\;&C\lambda\,(1+\lambda\,h)^2\,(1+\lambda L)\,\rho\,\|\{D_{\boldsymbol{u},p}F_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)\}^{-1}\|_{\mathscr{L}(\boldsymbol{H}_h\times Q_h)}\,|||(\boldsymbol{u}_h,p_h)-(\boldsymbol{v}_h,q_h)|||.
\end{aligned}\tag{4.24}$$

We are ready to fix parameter ρ, which is defined as follows

$$\rho:=\|\{D_{\boldsymbol{u},p}F(\lambda,\boldsymbol{u}_\lambda,p_\lambda)\}^{-1}\|_{\mathscr{L}(\boldsymbol{H}\times Q)}^{-1}\,|||F_h(\lambda,\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda)|||\,.$$

Next, replacing it in (4.24), and using lemmas 4.1 and 4.2, we arrive at

$$|||\Phi(\boldsymbol{u}_h,p_h)-\Phi(\boldsymbol{v}_h,q_h)|||\leqslant C(L,\Lambda)\,h\,|||(\boldsymbol{u}_h,p_h)-(\boldsymbol{v}_h,q_h)|||\,.\tag{4.25}$$

As a result, picking h small enough, Φ becomes a contraction from $B((\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda),\rho)$ onto itself, and we conclude that for $\lambda\in\Lambda$ fix, problem (3.10) has a unique solution $(\boldsymbol{u}_{h,\lambda},p_{h,\lambda})\in B((\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda);\rho)$. The error estimate follows using the triangle inequality

$$|||(\boldsymbol{u}_\lambda-\boldsymbol{u}_{h,\lambda},p_\lambda-p_{h,\lambda})|||\leqslant|||(\boldsymbol{u}_\lambda-\mathscr{I}_h\boldsymbol{u}_\lambda,p_\lambda-\mathscr{J}_h p_\lambda)|||+|||(\mathscr{I}_h\boldsymbol{u}_\lambda-\boldsymbol{u}_{h,\lambda},\mathscr{J}_h p_\lambda-p_{h,\lambda})|||,$$

and the fact that $(\boldsymbol{u}_{h,\lambda},p_{h,\lambda})\in B((\mathscr{I}_h\boldsymbol{u}_\lambda,\mathscr{J}_h p_\lambda);\rho)$ with $\rho\leqslant C(L,\Lambda)\,h$, and (2.9)–(2.14). □

5. Numerical validation

The exact solution of local problem (3.1) is (in general) not available, and then, method (3.3) requests a two-level discretization strategy. Nevertheless, an one-level version of method (3.3) can be proposed and still preserves the overall properties of the original two-level method. To see this more clearly, notice first that for a given function $\boldsymbol{v}=\nabla q\in L^2(K)^d$, the solution of problem (3.1) reads

$$\boldsymbol{u}_e^K(\nabla q)=\mathbf{0}\quad\text{and}\quad p_e^K(\nabla q)=\chi_h(q).\tag{5.1}$$

Also, observe that from (5.1), we can get the following closed formula

$$\begin{aligned} p_e^K(\Pi_K((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h)) &= \chi_h(\boldsymbol{x}\cdot(\nabla \boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h)\,, &(5.2)\\ p_e^K(\Pi_K((\nabla \boldsymbol{v}_h)\boldsymbol{u}_h)) &= \chi_h(\boldsymbol{x}\cdot(\nabla \boldsymbol{v}_h)\Pi_K\boldsymbol{u}_h)\,. &(5.3)\end{aligned}$$

As a result, we can simplify (3.3) by replacing $p_e^K((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h)$ by $p_e^K(\Pi_K((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h))$, and $p_e^K((\nabla \boldsymbol{v}_h)\boldsymbol{u}_h)$ by $p_e^K(\Pi_K((\nabla \boldsymbol{v}_h)\boldsymbol{u}_h))$, and using the (5.2) and (5.3) in method (3.3). The result is an one-level LPS method which reads: *Find* $(\boldsymbol{u}_h,\tilde{p}_h)\in \boldsymbol{H}_h\times Q_h$ *such that*

$$\begin{aligned} &\nu(\nabla \boldsymbol{u}_h,\nabla \boldsymbol{v}_h)+((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h,\boldsymbol{v}_h)-(\tilde{p}_h,\nabla\cdot\boldsymbol{v}_h)+(\tilde{q}_h,\nabla\cdot\boldsymbol{u}_h)\\ &+\sum_{K\in\mathscr{T}_h}\frac{\alpha_K}{\nu}\left[(\chi_h(\tilde{p}_h),\chi_h(\tilde{q}_h))_K+(\chi_h(\boldsymbol{x}\cdot(\nabla \boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h),\chi_h(\boldsymbol{x}\cdot(\nabla \boldsymbol{v}_h)\Pi_K\boldsymbol{u}_h))_K\right]\\ &+\sum_{K\in\mathscr{T}_h}\frac{\gamma_K}{\nu}\,(\chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{u}_h),\chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{v}_h))_K+\sum_{F\in\mathscr{E}_h}\tau_F\,([\![\tilde{p}_h]\!],[\![\tilde{q}_h]\!])_F=(\tilde{\boldsymbol{f}},\boldsymbol{v}_h)\,. \qquad (5.4)\end{aligned}$$

REMARK 5.1 One-level method (5.4) is more suitable to be implemented than the original LPS method (3.3) as the former avoids two-level computations. As such, version (5.4) will be adopted to perform numerical validations. Notice that the theoretical results developed for method (3.3) remains valid since the operator $p_e^K\circ\Pi_K$ shares the same property as the operator p_e^K (see the stability result in Lemma 3.1, for instance). This is a simple consequence of the L^2 stability of the projection operator Π_K. Consequently, overall theoretical results can be extended, with minor differences, to the simplified LPS method (5.4).

Next, numerical experiments validate the LPS method. We focus on three-dimensional test cases since similar conclusions may be also obtained from the two-dimensional tests. We first propose an analytical numerical test from which theoretical convergence results are validated. Then, we address numerical comparisons with some well-documented benchmarks from the literature.

5.1 *Analytic solution*

In this case $\Omega :=]0,1[^3$, $\nu = 1,\ 10^{-2}$. The right hand-side $\tilde{\boldsymbol{f}}$ is chosen such that the solution of the problem is given by

$$\boldsymbol{u}(x,y,z)=(e^x\sin z,\,-e^x\sin z,\,e^x\cos z-e^x\cos y)\quad\text{and}\quad p(x,y,z)=-\frac{1}{2}e^{2x}+\frac{1}{4}(e^2-1).$$

Figures 1–4 highlight the convergence rates for the finite elements $\mathbb{P}_1\times\mathbb{P}_1$ (with continuos pressure) and $\mathbb{P}_1\times\mathbb{P}_0$. We observe a perfect agreement with the theoretical results.

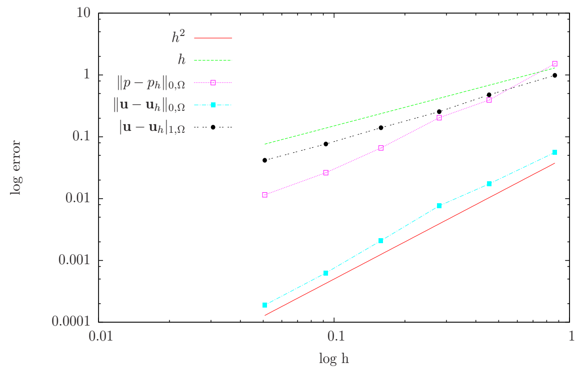

FIG. 1. Convergence history for the $\mathbb{P}_1 \times \mathbb{P}_1$ scheme ($\nu = 1$).

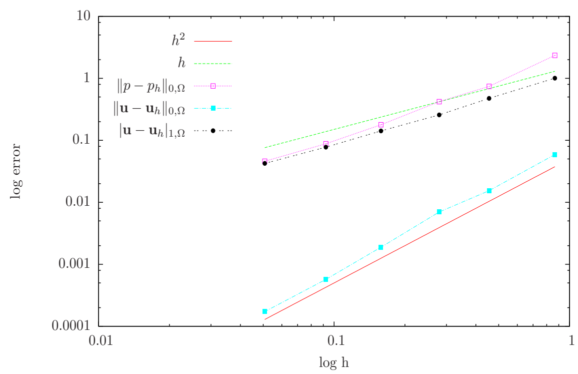

FIG. 2. Convergence history for the $\mathbb{P}_1 \times \mathbb{P}_0$ scheme ($\nu = 1$).

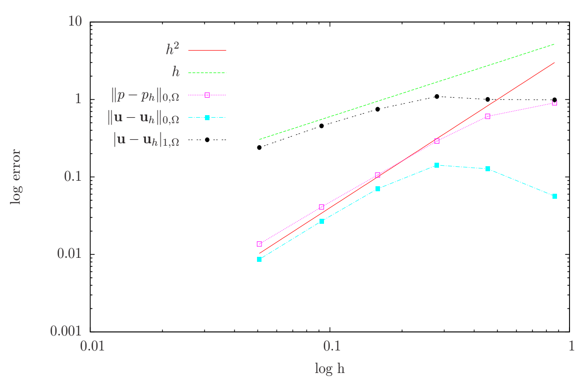

FIG. 3. Convergence history for the $\mathbb{P}_1 \times \mathbb{P}_1$ scheme ($\nu = 10^{-2}$).

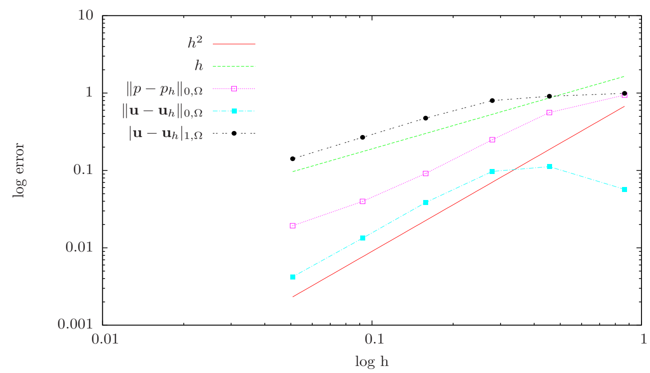

FIG. 4. Convergence history for the $\mathbb{P}_1 \times \mathbb{P}_0$ scheme ($\nu = 10^{-2}$).

5.2 *3D Lid–driven cavity problem*

We consider the rectangular cavity depicted in Figure 5 with the Dirichlet boundary conditions shown also in Figure 5. The Reynolds's number ($Re := \dfrac{1}{\nu}$) is set to $Re = 1500$.

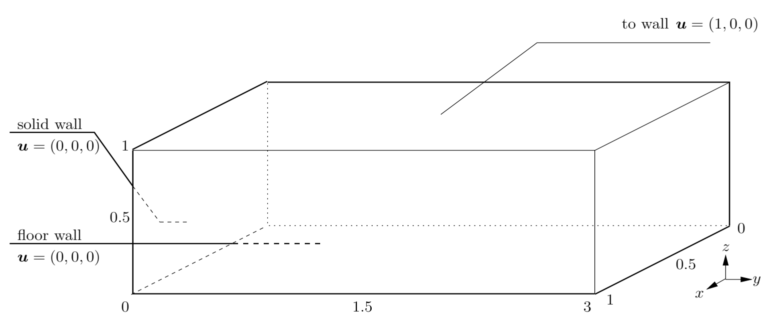

FIG. 5. Boundary conditions for the 3D lid–driven cavity.

Figures 6 and 9 show the trajectory of some particles released from $(0.5, 1.9, 0.03)$ and $(0.188, 1.567, 0.0708)$ and computed using the LPS method with the finite elements $\mathbb{P}_1 \times \mathbb{P}_1$ (continuos pressure) and $\mathbb{P}_1 \times \mathbb{P}_0$. We note that we recover the swirling paths which is characteristic of the three-dimensional flow (see Shankar & Deshpande (2000) and Chiang *et al.* (1996), for instance).

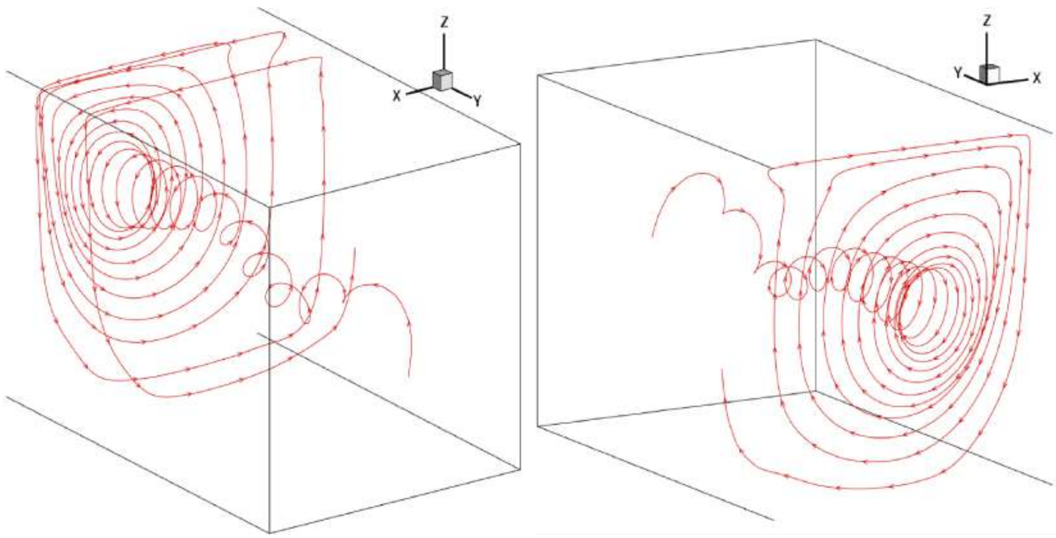

FIG. 6. Trajectory of particles released at $(0.5, 1.9, 0.03)$ (left) and at $(0.188, 1.567, 0.0708)$ (right) using the $\mathbb{P}_1 \times \mathbb{P}_1$ element. Here we picture only a half of the cavity.

Next, we depict in Figures 7 and 10 the streamlines on three different plane sections (set at inflow, at the middle and at the outflow of the domain) obtained using the finite elements $\mathbb{P}_1 \times \mathbb{P}_1$ and $\mathbb{P}_1 \times \mathbb{P}_0$. We observe a qualitative agreement with the results presented in Shankar & Deshpande (2000) and Chiang *et al.* (1996).

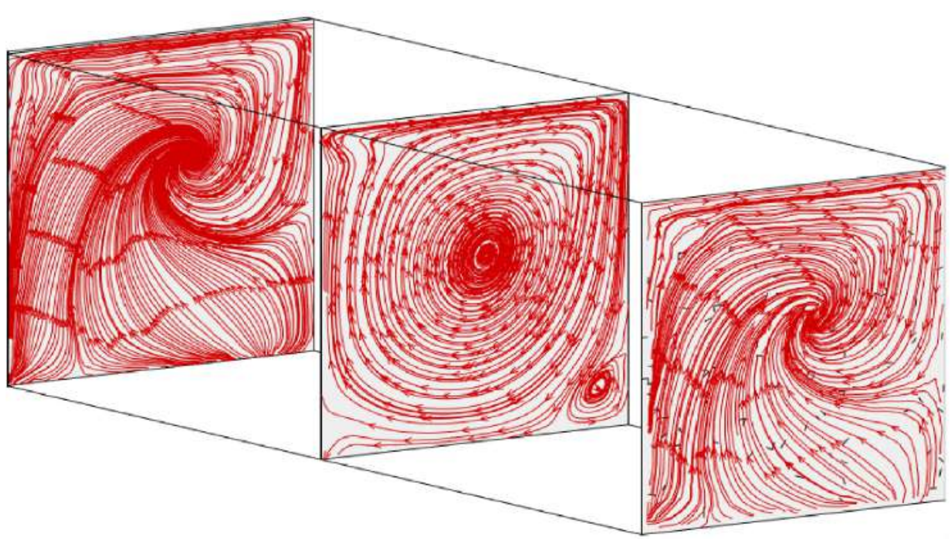

FIG. 7. Streamlines at different cross-sections using the finite element $\mathbb{P}_1 \times \mathbb{P}_1$.

Figures 8 and 11 shows the isosurfaces of the magnitude of the velocity field with value equal to 0.13, and the isosurfaces of the vorticity magnitude with value equal to 1. Such results are obtained using the finite elements $\mathbb{P}_1 \times \mathbb{P}_1$ and $\mathbb{P}_1 \times \mathbb{P}_0$. We observe the primary vortex within the cavity as expected, which turns to be qualitatively similar to the one presented in Ravnik *et al.* (2009).

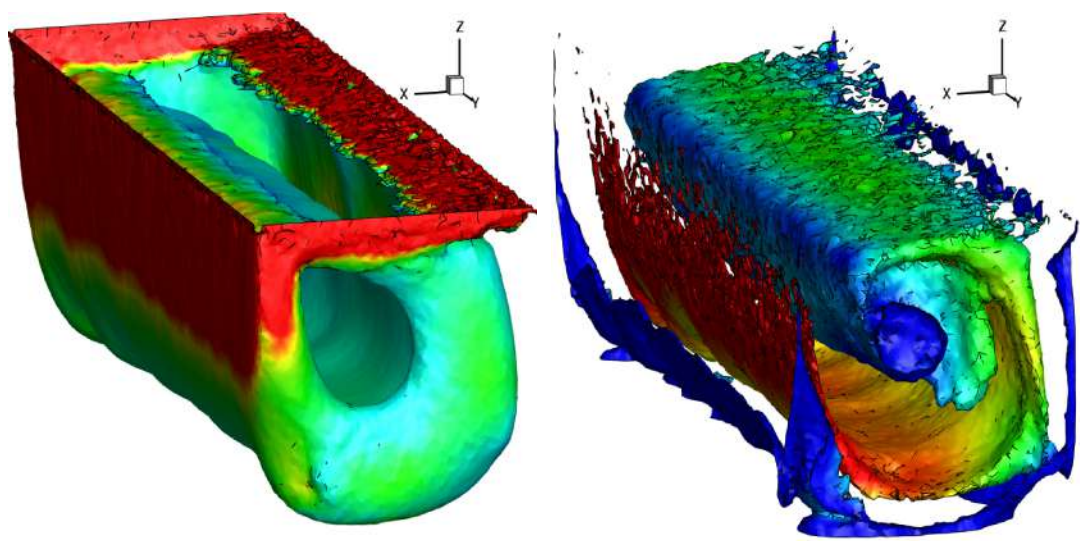

FIG. 8. Isosurface of $|\boldsymbol{u}| = 0.13$ (left) and isosurface of $|\nabla \times \boldsymbol{u}| = 1$ (right) with the $\mathbb{P}_1 \times \mathbb{P}_1$ element.

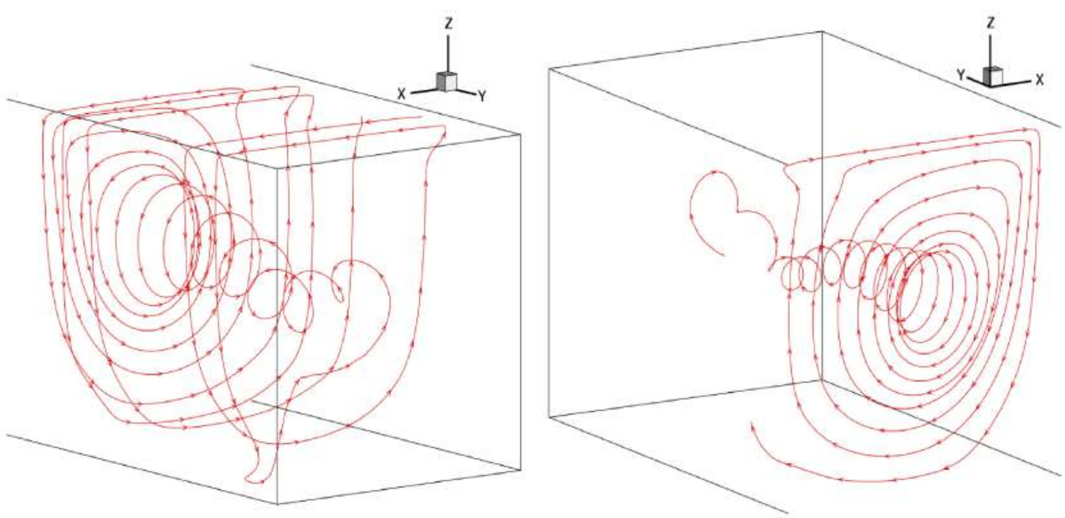

FIG. 9. Trajectory of particles released at $(0.5, 1.9, 0.03)$ (left) and at $(0.188, 1.567, 0.0708)$ (right) using the $\mathbb{P}_1 \times \mathbb{P}_0$ element. Here we picture only a half of the cavity.

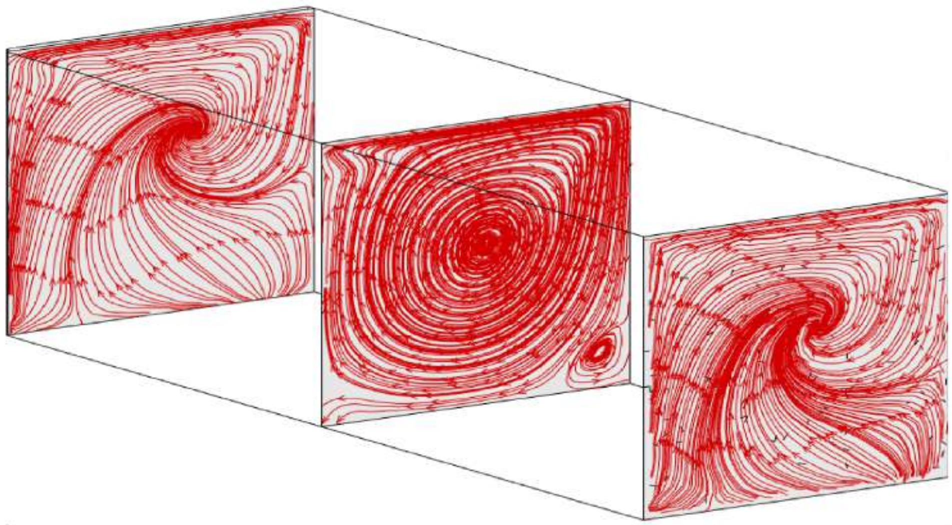

FIG. 10. Streamlines at different cross-sections using the finite element $\mathbb{P}_1 \times \mathbb{P}_0$.

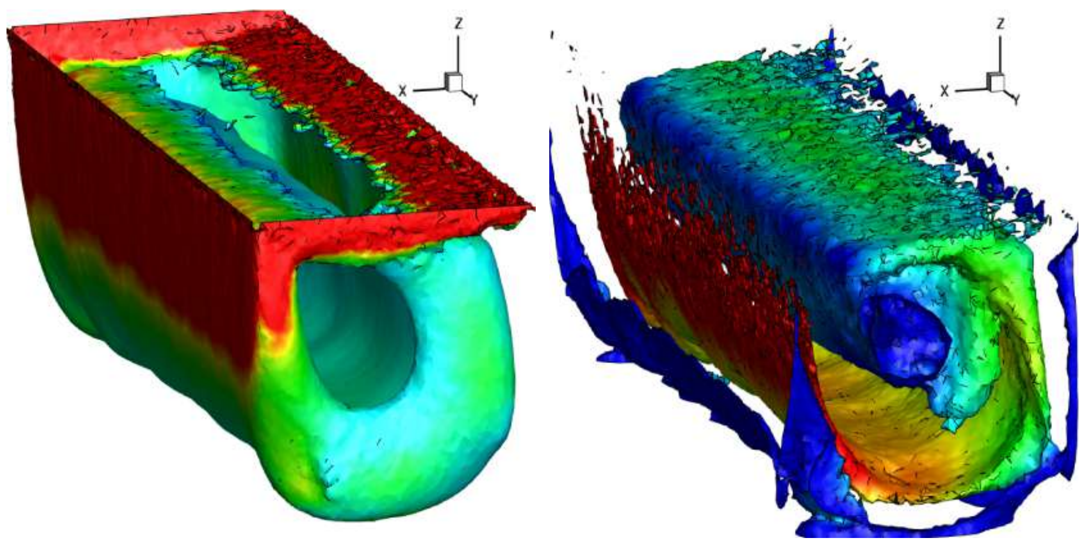

FIG. 11. Isosurface of $|\boldsymbol{u}| = 0.13$ (left) and isosurface of $|\nabla \times \boldsymbol{u}| = 1$ (right) with the $\mathbb{P}_1 \times \mathbb{P}_1$ element.

5.3 *The circular cylinder problem*

The statement of the problem is shown in Figure 12. The inflow velocity field is $\boldsymbol{u}_p = (7.2\,yz(0.41 - y)(0.41 - z)/0.41^4, 0, 0)^T$ and $\nu = 10^{-3}$ (for further details, see Turek (1999)).

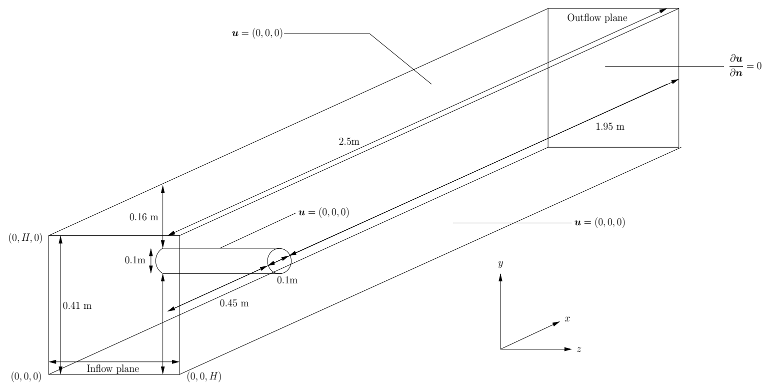

FIG. 12. Domain and boundary conditions for the flow problem around a cylinder with circular cross–section.

We validate our numerical results using the *drag* and the *lift* coefficients defined as follows

$$C_D := \frac{2}{\bar{u}^2 D 0.41} \int_S (\nu \frac{\partial \boldsymbol{v}_t}{\partial \boldsymbol{n}} n_y - p n_x)\, dS, \qquad C_L := -\frac{2}{\bar{u}^2 D 0.41} \int_S (\nu \frac{\partial \boldsymbol{v}_t}{\partial \boldsymbol{n}} n_x + p n_y)\, dS.$$

Here S is the boundary of the cylinder, $\boldsymbol{n} := (n_x, n_y, n_z)$ is the outward normal vector on S, $\boldsymbol{v}_t$ is the tangential component of the velocity on S and $\boldsymbol{t}$ is one of the tangent vectors on S. We set D, the cylinder diameter, to 0.1 and $\bar{u}$, the mean velocity, to $\frac{4}{9} u_p(0, 0.205, 0.205)$. Finally, we denote the difference of pressure between two points by

$$\Delta p := p(x_a, y_a, z_a) - p(x_e, y_e, z_e),$$

and we evaluate it at the points $(x_a, y_a, z_a) = (0.45, 0.2, 0.205)$ and $(x_e, y_e, z_e) = (0.55, 0.2, 0.205)$.

In Table 1 we compare the results from the LPS method with the ones obtained from other approaches in the literature. We see that the results remain at the same range of values (despite the differences at the order of approximation among the methods).

Scheme	C_D	C_L	Δp
Schäfer et. al. Schäfer & Turek (1996)	6.1295	0.0093	0.1693
ANSYS-CFX Bayraktar *et al.* (2012)	6.1829	0.0094	-
OpenFOAM Bayraktar *et al.* (2012)	6.1893	0.0097	-
FEATFLOW Becker & Braack (2001)	6.1847	0.0094	-
LPS $\mathbb{P}_1 \times \mathbb{P}_1$	6.0568	0.0088	0.1755
LPS $\mathbb{P}_1 \times \mathbb{P}_0$	5.9322	0.0084	0.1669

Table 1. The drag and lift coefficients, and the pressure obtained from different numerical methods.

Figures 13 and 14 show a zoom of the streamtraces and the isovalues of the magnitude of the velocity and of the pressure at the cross-section $z = 0.205$. Here we used the finite elements $\mathbb{P}_1 \times \mathbb{P}_1$ (continuos pressure) and $\mathbb{P}_1 \times \mathbb{P}_0$. Observe that the overall results are in accordance with the expected behavior of the fluid flow (see Bayraktar *et al.* (2012), for instance).

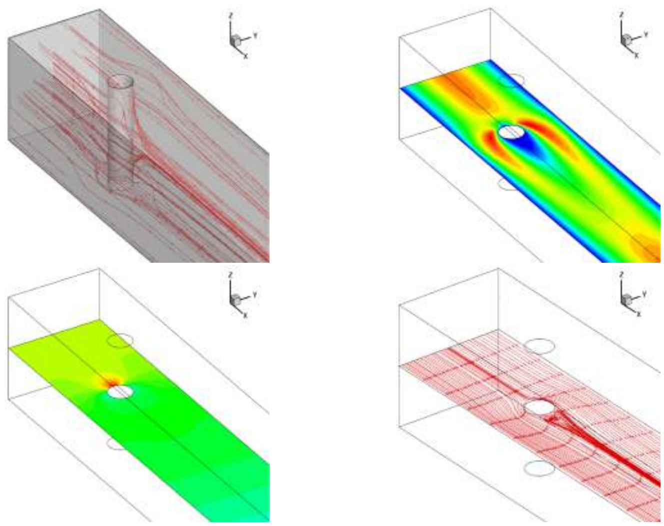

FIG. 13. The streamtracers (top left) and the isovalues of the magnitude of the velocity field (top right), and the isovalues of the magnitude of the pressure (bottom left) and streamlines of the velocity at a horizontal cross-section (bottom right). Here we used the finite element $\mathbb{P}_1 \times \mathbb{P}_1$.

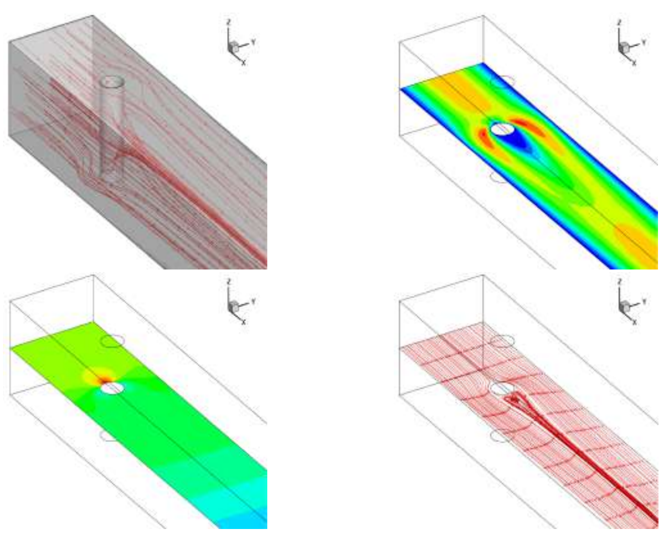

FIG. 14. The streamtracers (top left) and the isovalues of the magnitude of the velocity field (top right), and the isovalues of the magnitude of the pressure (bottom left) and streamlines of the velocity at a horizontal cross-section (bottom right). Here we used the finite element $\mathbb{P}_1 \times \mathbb{P}_0$.

6. Conclusions

We proposed a stabilized "term-by-term" finite element method for the fully non-linear incompressible Navier-Stokes equations. Driven by the solution of local Stokes problems, the stabilized terms (with their stabilization parameters) made the simplest element and the equal-order piecewise linear spaces inf–sup stable. From a practical standpoint, such an upscaling procedure has been performed analytically avoiding more involved two-level computations. As a result, it can be seen as a member of the LPS class of methods with no extra computational costs involved in the computation of the stabilized terms. Also, the method was proved to be optimal convergence despite of its lack of consistency. We conclude that the proposed LPS method is a competitive option to handle complex (eventually multiscale) flows, while remains a fair compromise between accuracy and computational cost as highlighted by the three-dimensional numerical validations.

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